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A SIMPLE STATISTIC TO TEST GENERALIZED PALINDROMIC SYMMETRY MODEL IN A 4 × 4 CONTINGENCY TABLE

SADAO TOMIZAWA*

Science University of Tokyo

For a 4×4 contingency table, this note gives a simple statistic to test the goodness-of-fit of the generalized palindromic symetry (GPS) model considered by McCullagh (1978). Also an asymptotic confidence interval for a parameter of interest in the GPS model is given. Two sets of unaided vision data are used as example.

Key words: Confidence interval; Cumulative odds ratio; Delta method; z–statistic.

1. INTRODUCTION

For the $R \times R$ square contingency table, let p_{ij} denote the probability that an observation will fall in the cell in row i and column $j\,(i = 1, \ldots, R; j = 1, \ldots, R)$. The generalized palindromic symmetry (GPS) model defined in McCullagh (1978) is given by

*Department of Information Sciences, Faculty of Sciences & Technology, Science University of Tokyo, Noda City, Chiba 278 Japan.
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$$
\begin{cases}
F_{ij} &= \exp\{\Delta_i/2\} \cdot \dfrac{\alpha_i}{\alpha_{j-1}}\phi_{ij} \quad (1 \le i < j \le R), \\
G_{ji} &= \exp\{-\Delta_i/2\} \cdot \dfrac{\alpha_{j-1}}{\alpha_i}\phi_{ji} \quad (1 \le i < j \le R), \\
p_{ii} &= \phi_{ii} \quad (1 \le i \le R),
\end{cases}
$$

where $\phi_{ij} = \phi_{ji}$, $\alpha_1 = 1$ and where

$$
F_{ij} = \sum_{s=1}^{i}\sum_{t=j}^{R} p_{st} \quad \text{and} \quad G_{ji} = \sum_{s=j}^{R}\sum_{t=1}^{i} p_{st}.
$$

Note that the GPS model is defined when $R \ge 4$. A special case of the GPS model obtained by putting $\Delta_1 = \Delta_2 = \cdots = \Delta_{R-1}$ is the original palindromic symmetry model considered in McCullagh (1978). And a further special case of the GPS model obtained by putting $\alpha_1 = \alpha_2 = \cdots = \alpha_{R-1}$ and $\Delta_1 = \Delta_2 = \cdots = \Delta_{R-1}$ is the conditional symmetry model (see McCullagh 1978 and Tomizawa 1989b).

Tomizawa (1989a, b) pointed out that the GPS model can be expressed as

$$
\Theta^U_{ij,st} = \Theta^L_{st,ij} \qquad (1 \le i < j < s < t \le R), \tag{1.1}
$$

where $\Theta^U_{ij} = F_{is}F_{jt}/(F_{js}F_{it})$ and $\Theta^L_{st,ij} = G_{si}G_{tj}/(G_{sj}G_{ti})$, being the odds ratios based on the $\{F_{ij}\}$ and $\{G_{ji}\}$. From (1.1), the GPS model states that for $1 \le i < j < s < t \le R$, if the odds that an observation is in row j or below rather than in row i or below is θ times higher when the observation is in column t or above rather than when it is in column s or above, then the odds that the observation is in column j or below rather than in column i or below is identically θ times higher when the observation is in row t or above rather than when it is in row s or above.

By the way, for testing the goodness-of-fit of the GPS model, the values of likelihood ratio chi-squared statistics G^2 and Pearson's chi-squared statistics χ^2 *could not* be calculated *easily* (even when $R = 4$) though the calculation would be possible, because *(i)* the model is not a multiplicative form, and *(ii)* the maximum likelihood estimates (MLEs) of expected frequencies cannot be written as a closed-form expression of the observations (see McCullagh 1978).

The purpose of this note is to give a simple statistic to test the goodness-of-fit of the GPS model when $R = 4$.

2. A SIMPLE TEST STATISTIC

Consider a 4×4 contingency table (i.e., $R = 4$). Then, from (1.1), the GPS model can be simply expressed as

$$\psi = 0 \qquad (\text{or } e^{\psi} = 1),$$

where

$$\psi = \log \Theta^{U}_{12,34} - \log \Theta^{L}_{34,12}.$$

Let n_{ij} denote the observed frequency in the cell (i, j) of the 4×4 table ($i = 1,2,3,4$; $j = 1,2,3,4$). Assuming that the $\{n_{ij}\}$ result from full multinomial sampling, we shall give a simple statistic to test the GPS model (i.e., $\psi = 0$), using the *delta method* of which descriptions are given by Bishop *et al.* (1975, Sec. 14.6) and Agresti (1984, p. 185, Appendix C). The sample version of ψ, i.e., $\hat{\psi}$, is given by ψ with $\{p_{ij}\}$ replaced by $\{\hat{p}_{ij}\}$, where $\hat{p}_{ij} = n_{ij}/n$ and $n = \Sigma\Sigma n_{ij}$. Using the delta method, $\sqrt{n}(\hat{\psi} - \psi)$ has asymptotically (as $n \to \infty$) a normal distribution with mean zero and variance

$$\begin{aligned} \sigma^2 \;=\; & \frac{1}{F_{14}} - \frac{1}{F_{13}} - \frac{1}{F_{23}} - \frac{1}{F_{24}} + \frac{2F_{14}}{F_{13}F_{24}} + \\ + \; & \frac{1}{G_{41}} - \frac{1}{G_{31}} - \frac{1}{G_{32}} - \frac{1}{G_{42}} + \frac{2G_{41}}{G_{31}G_{42}}. \end{aligned}$$

Let $\hat{\sigma}^2$ denote σ^2 with $\{p_{ij}\}$ replaced by $\{\hat{p}_{ij}\}$. When the GPS model holds true, $z = \sqrt{n}\hat{\psi}/\hat{\sigma}$ has asymptotically (as $n \to \infty$) the standard normal distribution; thus z^2 has asymptotically (as $n \to \infty$) a chi-squared distribution with one degree of freedom (df). In addition, the term $\hat{\sigma}/\sqrt{n}$ is an estimated approximate standard error for $\hat{\psi}$, and $\hat{\psi} \pm z_{p/2}\hat{\sigma}/\sqrt{n}$ is an approximate $100(1-p)$ percent confidence interval of ψ, where $z_{p/2}$ is the percentage point from the standard normal distribution corresponding to a two-tail probability equal to p.

3. EXAMPLES

Example 1: Table 1 is constructed from the data on unaided distance vision of 7477 women aged 30-39 employed in Royal Ordnance factories in Britain from 1943 to 1946. These data have been analyzed by many statisticians using various statistical models and methods; see, for example, Stuart (1955), Bishop *et al.* (1975, p. 284), McCullagh (1978), Goodman (1979), Tomizawa (1989a, b).

When the GPS model is applied to the data in Table 1, the value of z^2 (given in Section 2) is 5.71 ($P < 0.025$) with 1 df. Therefore this value is significant at 5% level. By the way, by calculating the MLEs of expected frequencies using the $G^2 = 6.18$ (from Tomizawa 1989a) and $\chi^2 = 6.15$ with 1 df. We can see now that the values of these three statistics are close.

Table 1

Unaided distance vision of British women; from Stuart (1955).

Right eye grade	Left eye grade				
	Highest	Second	Third	Lowest	Total
Highest	1520	266	124	66	1976
Second	234	1512	432	78	2256
Third	117	362	1772	205	2456
Lowest	36	82	179	492	789
Total	1907	2222	2507	841	7477

Next, for these data, the estimated value of ψ is $\hat{\psi} = -0.350$. The estimated approximate standard error of $\hat{\psi}$ is 0.146, and an approximate 95% confidence interval for ψ is $(-0.636, -0.063)$. [The corresponding confidence interval for $e^{\psi} = \Theta^U_{12,34}/\Theta^L_{34,12}$ is $(e^{-0.636}, e^{-0.063})$, or $(0.529, 0.939)$.] Since this interval for ψ does not contain the value zero, this would indicate that $\Theta^U_{12,34}$ is not equal (rather than equal) to $\Theta^L_{34,12}$.

Example 2: Table 2 is constructed from the data on unaided distance vision of 4746 students aged 18 to about 25 including women of about 10% in Faculty of Science and Technology, Science University of Tokyo in Japan examined in April 1982. These data have been analyzed earlier by Tomizawa (1984, 1989a).

When the GPS model is applied to the data in Table 2, the value of z^2 is 1.45 ($P > 0.2$) with 1 d.f. Also the values of G^2 and χ^2 are both 1.47 (G^2 value is taken directly from Tomizawa 1989a). Therefore the values of three statistics are quite close. (See Tomizawa 1989a for the interpretation obtained under the GPS model).

Table 2

Unaided distance vision of students in Japan; from Tomizawa (1984).

Right eye grade	Left eye grade				Total
	Highest	Second	Third	Lowest	
Highest	1291	130	40	22	1483
Second	149	221	114	23	507
Third	64	124	660	185	1033
Lowest	20	25	249	1429	1723
Total	1524	500	1063	1659	4746

Next, for these data, the estimated value of ψ is $\hat{\psi} = -0.241$. The estimated approximate standard error of $\hat{\psi}$ is 0.200, and an approximate 95% confidence interval for ψ is $(-0.633, 0.151)$. [The corresponding confidence interval for e^{ψ} is $(e^{-0.633}, e^{0.151})$, or $(0.531, 1.163)$.] Since this interval for ψ contains the value zero, this would indicate that $\Theta^U_{12,34}$ is equal to $\Theta^L_{34,12}$, or even if $\Theta^U_{12,34}$ is not equal to $\Theta^L_{34,12}$, the degree of non-equality between two odds ratios is slight.

4. REMARK

As seen in Sections 2 and 3, the z^2–statistic (or z–statistic) has the advantage such that it can be easily calculated without the use of computer. Therefore, if one wants to check whether the GPS model fits well or poorly for a 4×4 table data, we recommend using the z^2–statistic (or z–statistic) given in this note.

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