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GOODNESS-OF-FIT TEST FOR THE FAMILY OF LOGISTIC DISTRIBUTIONS

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Chi-squared goodness-of-fit test for the family of logistic distributions is proposed. Different methods of estimation of the unknown parameters θ of the family are compared. The problem of homogeneity is considered.

Key words: Logistic distribution; BAN estimator; method of moments; maximum likelihood method; Chernoff-Lehmann theorem; MVUE; homogeneity problem; chi-squared test.

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1. INTRODUCTION

Let $X_1, \ldots, X_n$ be independent identically distributed random variables and suppose that according to the hypothesis H_0

$$(1)\quad \mathbf{P}\{X_i \le x\} = F(x;\boldsymbol{\theta}), \qquad \boldsymbol{\theta} = (\theta_1, \ldots, \theta_s)^T \in \Theta \subset \mathbb{R}^s, \qquad x \in \mathbb{R}^1,$$

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where Θ is an open set. We devide the real line into k intervals $I_1, \ldots, I_k$:

$$I_1 \cup \cdots \cup I_k = \mathbb{R}^1, \qquad I_i \cap I_j = \emptyset, \quad i \neq j.$$

We shall suppose that

$$p_i(\boldsymbol{\theta}) = \mathbf{P}\{X_1 \in I_i \mid H_0\} > 0, \quad i = 1, \ldots, k. \tag{2}$$

Let $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_k)^T$ be the vector of frequencies arising as a result of grouping the random variables $X_1, \ldots, X_n$ into the classes $I_1, \ldots, I_k$. We denote

$$X_n^2(\boldsymbol{\theta}) = \mathbf{X}_n^T(\boldsymbol{\theta})\mathbf{X}_n(\boldsymbol{\theta}) = \sum_{i=1}^{k} \frac{(\nu_i - np_i(\boldsymbol{\theta}))^2}{np_i(\boldsymbol{\theta})}, \tag{3}$$

where

$$\mathbf{X}_n = \left(\frac{\nu_1 - np_1(\boldsymbol{\theta})}{\sqrt{np_1(\boldsymbol{\theta})}}, \ldots, \frac{\nu_k - np_k(\boldsymbol{\theta})}{\sqrt{np_k(\boldsymbol{\theta})}} \right)^T. \tag{4}$$

Theorem. (K. PEARSON, 1900)

If $\boldsymbol{\theta}$ is known or given by the hypothesis H_0, then

$$\lim_{n \to \infty} \mathbf{P}\{X_n^2(\boldsymbol{\theta}) \geq x \mid H_0\} = \mathbf{P}\{\chi_{k-1}^2 \geq x\} \tag{5}$$

It is known that if the value of the parameter $\boldsymbol{\theta}$ is unknown and is estimated relative to the observed values of $X_1, \ldots, X_n$, the limiting distribution of the Pearson's statistics $X_n^2(\boldsymbol{\theta}_n^*)$ is given by the asymptotic properties of the estimator $\boldsymbol{\theta}_n^*$ which is substituted into (3) in place of $\boldsymbol{\theta}$.

Here we shall give some results concerning this problem.

2. FISHER'S THEOREM

Following Cramer (1946) we suppose that

1) $p_i(\boldsymbol{\theta}) > c > 0$, $i = 1, \ldots, k$, $(k \geq s + 2)$;

2) $\frac{\partial^2 p_i(\boldsymbol{\theta})}{\partial \theta_j \partial \theta_l}$ are continuous functions;

3) the information matrix of Fisher

$$\mathbf{J} = \mathbf{J}(\boldsymbol{\theta}) = \left\| \sum_{l=1}^{k} \frac{1}{p_l(\boldsymbol{\theta})} \frac{\partial p_l(\boldsymbol{\theta})}{\partial \theta_i} \frac{\partial p_l(\boldsymbol{\theta})}{\partial \theta_j} \right\|_{s\times s} = \mathbf{B}^T(\boldsymbol{\theta})\mathbf{B}(\boldsymbol{\theta}) \tag{6}$$

exists, and rank$\mathbf{J} = s$, where

$$\mathbf{B}(\boldsymbol{\theta}) = \left\| \frac{1}{\sqrt{p_l(\boldsymbol{\theta})}} \frac{\partial p_l(\boldsymbol{\theta})}{\partial \theta_j} \right\|_{k\times s}. \tag{7}$$

In this case $n\mathbf{J}$ is the information matrix of Fisher of the statistic $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_k)^T$.

Let $\tilde{\boldsymbol{\theta}}_n$ is the minimum chi-squared estimator for $\boldsymbol{\theta}$,

$$X_n^2(\tilde{\boldsymbol{\theta}}_n) = \min_{\boldsymbol{\theta}\in\Theta} X_n^2(\boldsymbol{\theta}), \tag{8}$$

or an estimator asymptotically equivalent to it. As it was shown by Cramer(1946), a root $\tilde{\boldsymbol{\theta}}_n$ of the system

$$\sum_{i=1}^{k} \frac{\nu_i}{np_i(\boldsymbol{\theta})} \frac{\partial p_i(\boldsymbol{\theta})}{\partial \theta_j} = 0, \qquad j = 1, \ldots, s. \tag{9}$$

is such an estimator and, under H_0 as $n \to \infty$, the vector $\sqrt{n}(\tilde{\boldsymbol{\theta}}_n - \boldsymbol{\theta})$ satisfies the asymptotic relation

$$\sqrt{n}(\tilde{\boldsymbol{\theta}}_n - \boldsymbol{\theta}) = \mathbf{J}^{-1}(\boldsymbol{\theta})\mathbf{B}^T(\boldsymbol{\theta})\mathbf{X}_n(\boldsymbol{\theta}) + \mathrm{o}(\mathbf{1}_s), \tag{10}$$

where $\mathrm{o}(\mathbf{1}_s)$ is a random vector converging to $\mathbf{0}_s$ in $\mathbf{P}_{\boldsymbol{\theta}}$ - probability, and hence $\sqrt{n}(\tilde{\boldsymbol{\theta}}_n - \boldsymbol{\theta})$ is asymptotically normally distributed with parameters $\mathbf{0}_s$ and $\mathbf{J}^{-1}(\boldsymbol{\theta})$.

Theorem. (FISHER(1928), CRAMER(1946))

If the regularity conditions of Cramer hold then

$$\lim_{n\to\infty} \mathbf{P}\{X_n^2(\tilde{\boldsymbol{\theta}}_n) \geq x \mid H_0\} = \mathbf{P}\{\chi^2_{k-s-1} \geq x\}. \tag{11}$$

3. CHERNOFF-LEHMANN'S THEOREM

We suppose that the regularity conditions of Chernoff-Lehmann's (1954) hold:

1) $F(x;\boldsymbol{\theta})$ has probability density $f(x;\boldsymbol{\theta})$, where all

$$\frac{\partial^2 f(x;\boldsymbol{\theta})}{\partial\theta_j\partial\theta_l} \tag{12}$$

are continuous functions on $\mathbf{R}^1 \times \Theta$;

2) the information matrix of Fisher

$$\mathbf{I}(\boldsymbol{\theta}) = \|I_{ij}\|_{s\times s} = \mathbf{E}_{\boldsymbol{\theta}}\boldsymbol{\Lambda}(\boldsymbol{\theta})\boldsymbol{\Lambda}^T(\boldsymbol{\theta}), \tag{13}$$

corresponding to one observation X_1 exists and is positive definite for any $\boldsymbol{\theta} \in \Theta$, where

$$\boldsymbol{\Lambda}(\boldsymbol{\theta}) = \frac{\partial}{\partial\boldsymbol{\theta}} \ln f(X_1;\boldsymbol{\theta}); \tag{14}$$

($n\mathbf{I}(\boldsymbol{\theta})$ is the amount of information of Fisher about $\boldsymbol{\theta}$ in sample $\mathbb{X} = (X_1, \ldots, X_n)^T$).

3) differentiation with respect to parameters under the integral sign of

$$\int f(x;\boldsymbol{\theta})dx = 1 \tag{15}$$

is permissible, i.e.

$$\frac{\partial}{\partial\theta_i}\int f(x;\boldsymbol{\theta})dx = \int \frac{\partial}{\partial\theta_i} f(x;\boldsymbol{\theta})dx = 0, \qquad i = 1, \ldots, s; \tag{16}$$

4) the matrix $\mathbf{W} = [w_{ij}]$ with elements

$$w_{ij} = \int_{I_j} \frac{\partial}{\partial\theta_i} f(x;\boldsymbol{\theta})dx \tag{17}$$

has rank s;

5) the maximum likelihood estimator $\hat{\boldsymbol{\theta}}_n$ exists,

$$L(\hat{\boldsymbol{\theta}}_n) = \sup_{\boldsymbol{\theta}\in\Theta} L(\boldsymbol{\theta}), \tag{18}$$

where

$$L(\boldsymbol{\theta}) = \prod_{i=1}^{n} f(X_i;\boldsymbol{\theta}). \tag{19}$$

As it is known (see for example Rao (1965)), $\hat{\boldsymbol{\theta}}_n$ is a solution of the likelihood equation

$$\boldsymbol{\Lambda}(\boldsymbol{\theta}) = \mathbf{0}_s, \tag{20}$$

and, under H_0 as $n \to \infty$, the vector $\sqrt{n}(\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta})$ satisfies the asymptotic relation

$$\sqrt{n}(\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}) = \frac{1}{\sqrt{n}}\mathbf{I}^{-1}(\boldsymbol{\theta})\boldsymbol{\Lambda}(\boldsymbol{\theta}) + o(\mathbf{1}_s), \tag{21}$$

from where it follows that $\sqrt{n}(\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta})$ is asymptotically normally distributed with parameters $\mathbf{0}_s$ and $\mathbf{I}^{-1}(\boldsymbol{\theta})$ and hence $\hat{\boldsymbol{\theta}}_n$ is asympotically efficient estimator. We say that $\hat{\boldsymbol{\theta}}_n$ is a BAN estimator.

Theorem. (Chernoff-Lehmann, 1954)

If the regularity conditions 1)-5) hold then

$$\lim_{n\to\infty} \mathbf{P}\{X_n^2(\hat{\boldsymbol{\theta}}_n) \geq x \mid H_0\} = \mathbf{P}\{\chi^2_{k-s-1} + \sum_{i=1}^{s} \lambda_i \xi_i^2 \geq x\}, \tag{22}$$

where $\chi^2_{r-s-1}, \xi_1, \ldots, \xi_s$ are independent, $\xi_i \sim N(0,1)$ and $\lambda_i = \lambda_i(\boldsymbol{\theta})$, $0 < \lambda_i(\boldsymbol{\theta}) < 1$, $i = 1, 2, \ldots, s$, are the roots of the equation

$$\mid (1-\lambda)\mathbf{I}(\boldsymbol{\theta}) - \mathbf{J}(\boldsymbol{\theta}) \mid = 0. \tag{23}$$

Remark 1. We note here that in continuous case $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_k)^T$ is not sufficient statistic, and hence the matrix $\mathbf{I}(\boldsymbol{\theta}) - \mathbf{J}(\boldsymbol{\theta})$ is positive definite.

Remark 2. Let us consider the density family

$$f(x;\boldsymbol{\theta}) = h(x)exp\{\sum_{m=1}^{s} \theta_m x^m + v(\boldsymbol{\theta})\}, \qquad x \in \mathcal{X} \subseteq \mathbb{R}^1, \tag{24}$$

$\mathcal{X}$ is open in $\mathbf{R}^1$, $\mathcal{X} = \{x : f(x;\boldsymbol{\theta}) > 0\}$, $\qquad \boldsymbol{\theta} \in \Theta$.

The family (24) is very rich: it contains Poisson, normal distributions etc. It's evident that

$$\mathbf{U}_n = \left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} X_i^2, \ldots, \sum_{i=1}^{n} X_i^s\right)^T \tag{25}$$

is complet minimal sufficient statistics for the family (24).

We suppose that

1) the support $\mathcal{X}$ does not depend on $\boldsymbol{\theta}$;

2) the matrix of Hessen

$$\mathbf{H}_v(\boldsymbol{\theta}) = -\left\| \frac{\partial^2}{\partial\theta_i \partial\theta_j} v(\boldsymbol{\theta}) \right\|_{s\times s} \tag{26}$$

of the function $v(\boldsymbol{\theta})$ is positive definite;

3) the moment $a_s(\boldsymbol{\theta}) = \mathbf{E}_{\boldsymbol{\theta}} X_1^s$ exists.

In this case, using the results of Zacks (1971), it is not difficult to show (see, for example, Dzhaparidze and Nikulin (1991)) that the maximum likelihood estimator $\hat{\boldsymbol{\theta}}_n = \hat{\boldsymbol{\theta}}_n(\mathbf{U}_n)$ and the method of moments estimator $\bar{\boldsymbol{\theta}}_n = \bar{\boldsymbol{\theta}}_n(\mathbf{U}_n)$ of $\boldsymbol{\theta}$ coincide, i.e. $\bar{\boldsymbol{\theta}}_n = \hat{\boldsymbol{\theta}}_n$. Let

$$\mathbf{a}(\boldsymbol{\theta}) = (a_1(\boldsymbol{\theta}), \ldots, a_s(\boldsymbol{\theta}))^T \qquad \text{and} \qquad \mathbf{T}_n = \frac{1}{n}\mathbf{U}_n.$$

One can verify that

$$\mathbf{a}(\boldsymbol{\theta}) = -\frac{\partial}{\partial \boldsymbol{\theta}} v(\boldsymbol{\theta}),$$

and hence the likelihood equation is $\mathbf{T}_n = \mathbf{a}(\boldsymbol{\theta})$, i.e. $\hat{\boldsymbol{\theta}}_n$ is root of this equation. On the other hand we have $\mathbf{E}_{\boldsymbol{\theta}}\mathbf{T}_n \equiv \mathbf{a}(\boldsymbol{\theta})$, and hence from the properties of the statistics $\mathbf{U}_n$ it follows that $\mathbf{T}_n$ is the **MVUE** of $\mathbf{a}(\boldsymbol{\theta})$, and $\bar{\boldsymbol{\theta}}_n$ is the root of the same equation $\mathbf{T}_n = \mathbf{a}(\boldsymbol{\theta})$, which we used to find $\hat{\boldsymbol{\theta}}_n$. Hence $\hat{\boldsymbol{\theta}}_n = \bar{\boldsymbol{\theta}}_n$, i.e. under the conditions 1)-3) the method of moments gives for the family (24) an asymptotically efficient (BAN) estimator. We remark that in general *an estimator based on the method of moments is not asymptotically efficient, and hence does not verify the Chernoff-Lehmann theorem.* In "Handbook of the logistic distribution" in chap. 13, it is reported that this theorem is applied by Massaro and d'Agostino using $\bar{\boldsymbol{\theta}}_n = (\bar{\mathbf{X}}_n, s_n^2)^T$, (the moments method estimator of $\boldsymbol{\theta} = (\mathbf{E}X_1, \mathbf{Var}X_1)^T$) for the family of the logistic distributions. But $\bar{\boldsymbol{\theta}}_n$ is not efficient and ever not asymptotically efficient for the logistic family, and hence is not BAN, since this family does not belong to the exponential family (24) and $(\bar{\mathbf{X}}_n, s_n^2)^T$ is not sufficient statistic in this situation. Hence, the tables of critical points, proposed by Massaro et d'Agostino in section 13.9 are not valid.

4. ROY'S EXTENSION OF THE CHERNOFF-LEHMANN THEOREM

We consider here the result of Dahiya and Gurland (1970,1972), concerning the chi-squared test of Pearson with random intervals, which is an extension of the unpublished result of Roy (1956) and Chernoff-Lehmann's theorem. It can be found more information about Chernoff-Lehmann's theorem in the paper of LeCam, Mahan and Singh (1983).

Let $\boldsymbol{\theta}_n^*$ is an $\sqrt{n}$ - consistent estimator for $\boldsymbol{\theta}$ such that

$$\sqrt{n}(\boldsymbol{\theta}_n^* - \boldsymbol{\theta}) = \frac{1}{n}\sum_{j=1}^{n} \mathbf{v}(X_j) + o(\mathbf{1}_s), \tag{27}$$

where a function $\mathbf{v} = (v_1, \ldots, v_s)^T$ is such that

$$\mathbf{E}_{\boldsymbol{\theta}}\mathbf{v}(X_1) = \mathbf{0}_s \qquad \text{and} \qquad \mathbf{Var}_{\boldsymbol{\theta}}\mathbf{v}(X_1) = \mathbf{V} \quad \text{is finite.} \tag{28}$$

As it follows from (10) and (21) $\tilde{\boldsymbol{\theta}}_n$ and $\hat{\boldsymbol{\theta}}_n$ satisfy (27). For each $\boldsymbol{\theta} \in \Theta$ let us define a partition of the real line into k classes-intervals $I_1, I_2, \ldots, I_k$ is defined with boundary points $x_0 = -\infty$, $x_1 = \gamma_1(\boldsymbol{\theta}_n^*)$,...,$x_{k-1} = \gamma_{k-1}(\boldsymbol{\theta}_n^*)$, $x_k = +\infty$ depending on $\boldsymbol{\theta}_n^*$ such that

$$I_i = I_i(\boldsymbol{\theta}_n^*) = \{x : \gamma_{i-1}(\boldsymbol{\theta}_n^*) \leq x < \gamma_i(\boldsymbol{\theta}_n^*)\}, \qquad i = 1, \ldots, k, \tag{29}$$

where $\gamma_i(\boldsymbol{\theta})$ are continuous functions having partial derivatives.

Let $\boldsymbol{\nu}^* = (\nu_1^*, \ldots, \nu_k^*)^T$ be a vector of frequencies obtained as a result of grouping the observations $X_1, \ldots, X_n$ by the intervals $I_1(\boldsymbol{\theta}_n^*), I_2(\boldsymbol{\theta}_n^*), \ldots, I_k(\boldsymbol{\theta}_n^*)$ with random boundaries, and let

$$\begin{aligned} p_i(\boldsymbol{\theta}_n^*) &= p_i(\boldsymbol{\theta}_n^*, \boldsymbol{\theta}_n^*) = \mathbf{P}_{\boldsymbol{\theta}_n^*}\{X_1 \in I_i(\boldsymbol{\theta}_n^*) \mid H_0\} = \\ &= = F(\gamma_i(\boldsymbol{\theta}_n^*); \boldsymbol{\theta}_n^*) - F(\gamma_{i-1}(\boldsymbol{\theta}_n^*); \boldsymbol{\theta}_n^*). \end{aligned} \tag{30}$$

To test H_0 Roy (1956) proposed to consider the statistic

$$X_R^2(\boldsymbol{\theta}_n^*) = X_n^2(\boldsymbol{\theta}_n^*, \boldsymbol{\theta}_n^*) = \sum_{i=1}^{k} \frac{(\nu_i^* - np_i(\boldsymbol{\theta}_n^*))^2}{np_i(\boldsymbol{\theta}_n^*)}. \tag{31}$$

Theorem. (Roy(1956), Dahiya & Gurland (1972))

If $\boldsymbol{\theta}_n^*$ satisfies (27) and the Chernoff-Lehmann conditions hold then

$$\lim_{n\to\infty} \mathbf{P}\{X_R^2(\boldsymbol{\theta}_n^*) \geq x \mid H_0\} = \mathbf{P}\{\sum_{i=1}^{k} \lambda_i \xi_i^2 \geq x\}, \tag{32}$$

where $\xi_1, \xi_2, \ldots, \xi_k$ are mutually independent standard normal random variables, $\xi_i \sim N(0, 1)$, $\lambda_1 = \lambda_1(\boldsymbol{\theta}), \ldots, \lambda_k = \lambda_k(\boldsymbol{\theta})$ are the characteristic roots of the matrix $\mathbf{D}^{-1}\boldsymbol{\Sigma}$, where $\mathbf{D} = \mathbf{D}(\boldsymbol{\theta})$ is the diagonal matrix with the elements $p_1(\boldsymbol{\theta}), \ldots, p_k(\boldsymbol{\theta})$ on the main diagonal,

$$\boldsymbol{\Sigma} = \boldsymbol{\Sigma}(\boldsymbol{\theta}) = \mathbf{D} - \mathbf{p}\mathbf{p}^T - \mathbf{U}^T\mathbf{M} - \mathbf{M}^T\mathbf{U} + \mathbf{U}^T\mathbf{V}\mathbf{U}, \tag{33}$$

$$\mathbf{U}^T = [U_{ij}]_{k\times s}, \qquad U_{ij} = U_{ij}(\boldsymbol{\theta}) = \int_{g_{i-1}(\boldsymbol{\theta})}^{g_i}(\boldsymbol{\theta}) \frac{\partial f(x;\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} dx,$$

$$\mathbf{M}^T = [M_{ij}]_{k\times s}, \qquad M_{ij} = M_{ij}(\boldsymbol{\theta}) = \int_{g_{i-1}(\boldsymbol{\theta})}^{g_i}(\boldsymbol{\theta}) v_j(\boldsymbol{\theta}) f(x;\boldsymbol{\theta}) dx.$$

For example, if $\boldsymbol{\theta}_n^* = \hat{\boldsymbol{\theta}}_n$ is the maximum likelihood estimator, which satisfies (21), then, as it was shown by Roy (1956), in (32) $k-s-1$ of λ_i are equal to 1, one of the λ_i is equal to 0, and the remaining s lie between 0 and 1. It is obvious that this is an extension of the Chernoff-Lehmann theorem to the case of random cell boundaries.

Remark 3. As it was shown by Dahiya and Gurland (1972) if the density function $f(x;\boldsymbol{\theta})$ of X_1 belongs to a location and scale family

$$f(x;\boldsymbol{\theta}) = \frac{1}{\sqrt{\theta_2}} f\left(\frac{x-\theta_1}{\sqrt{\theta_2}}\right), \qquad \boldsymbol{\theta} = (\theta_1, \theta_2)^T, \qquad |\theta_1| < \infty, \theta_2 > 0, \tag{34}$$

then it is possible to choose the grouping intervals in order to have the asymptotic distribution of X_R^2 independent of $\boldsymbol{\theta}$. For example, let suppose that we test hypothesis H_0 according to which X_i follows the normal distribution $N(\theta_1, \theta_2)$,

$$\mathbf{E}\{X_i \mid H_0\} = \theta_1, \qquad \mathbf{Var}\{X_i \mid H_0\} = \theta_2, \tag{35}$$

and let $\bar{\boldsymbol{\theta}}_n = (\bar{X}_n, s_n^2)^T$, where

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i, \qquad s_n^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2. \tag{36}$$

$\bar{\boldsymbol{\theta}}_n$ is the method of moments estimator for $\boldsymbol{\theta} = (\theta_1, \theta_2)^T$. Since $f(x;\boldsymbol{\theta})$ belongs to the exponential family (24) of order 2, $s = 2$, the method of moments gives as a result the maximum likelihood estimator, $\hat{\boldsymbol{\theta}}_n = \bar{\boldsymbol{\theta}}_n$.

If in (29) we choose

$$\gamma_i(\boldsymbol{\theta}) = \theta_1 + c_i\sqrt{\theta_2}, \tag{37}$$

and hence $\gamma_i(\hat{\boldsymbol{\theta}}_n) = \bar{X}_n + c_i s_n$, then as it was shown by Gurland and Dahiya (1972) (see also Watson (1957),(1958)), the statistic X_R^2 is distributed, under H_0, in the limit as $n \to \infty$ as

$$\chi_{k-3}^2 + \lambda_1\xi_1^2 + \lambda_2\xi_2^2, \tag{38}$$

where λ_1 and λ_2 do not depend on $\boldsymbol{\theta}$. Using some results of Watson (1957), Dahiya and Gurland tabulated the distribution of Roy's statistic X_R^2 for $k = 3, 4, \ldots, 15$, for the significance level α equal to 0.1,0.05,0.01, in the case when the constants c_i in (37) are chosen so that $p_i(\hat{\boldsymbol{\theta}}_n) = 1/k$.

Remark 4. It is important to remark that when $\boldsymbol{\theta}$ is unknown and we have to estimate it, the limit distribution of the Pearson's statistic $X_n^2(\boldsymbol{\theta}_n^*)$ changes in general in accordance with asymptotical properties of the estimator $\boldsymbol{\theta}_n^*$ we shall use.

For this reason it has looked reasonable to have a statistic which limit distribution is wellknown when we apply the maximum likelihood estimator or anyone BAN estimator. In the papers of Nikulin (1973) (see also, for example, Rao and Robson (1974), Moore and Spruill (1975)), is exposed how to construct a chi-squared test for a continuous distribution (in particular, for the normal distribution and for distributions with shift and scale parameters), based on the statistic $Y_n^2(\boldsymbol{\theta}_n^*)$, by using any BAN estimator $\boldsymbol{\theta}_n^*$ of $\boldsymbol{\theta}$. For example we can take $\boldsymbol{\theta}_n^* = \hat{\boldsymbol{\theta}}_n$, where $\hat{\boldsymbol{\theta}}_n$ is the maximum likelihood estimator. We note that the technique of chi-squared tests for the exponential family of distributions of rank one, s=1, and some applications of MVUE's were exposed by Nikulin and Voinov (1989). Another modification $W_n^2(\boldsymbol{\theta}_n^*)$, which limit distribution is stable with respect to any statistical method of estimation, providing $\sqrt{n}$ - consistent estimator $\boldsymbol{\theta}_n^*$ was proposed by Dzhaparidze and Nikulin (1974). We shall apply the statistic Y_n^2 to test the hypothesis H_0 according to which the distribution of $\mathbf{X}_i$ belongs to the family of the logistic distributions. This topic is studied also by Dudley (1976), Drost(1988).

5. LOGISTIC DISTRIBUTION AND THE CHI-SQUARED GOODNESS-OF-FIT TEST

Let $\mathbb{X} = (X_1, \ldots, X_n)^T$ be a random sample, i.e. $X_1, \ldots, X_n$ are independent identically distributed random variables. In this section we consider the problem of testing the hypothesis H_0 that the distribution function of X_1 belongs to the family of logistic distributions $G\left(\frac{x-\mu}{\sigma}\right)$ depending on the shift parameter μ and the scale parameter σ:

$$(39)\ \mathbf{P}\{X_1 \le x \mid H_0\} = G\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{1+\exp\{-\frac{\pi}{\sqrt{3}}\left(\frac{x-\mu}{\sigma}\right)\}}, \qquad x \in \mathbb{R}^1,$$

$$\mu = \mathbf{E}\{X_1 \mid H_0\}, \qquad |\mu| < \infty, \qquad \sigma^2 = \mathbf{Var} X_1, \qquad \sigma > 0.$$

Under H_0 the density function of X_i is

$$(40)\quad \frac{1}{\sigma} g\left(\frac{x-\mu}{\sigma}\right) = G'\left(\frac{x-\mu}{\sigma}\right) = \frac{\pi}{\sqrt{3}\sigma} \frac{exp\left(-\frac{\pi}{\sqrt{3}}\frac{x-\mu}{\sigma}\right)}{\left[1+exp\left(-\frac{\pi}{\sqrt{3}}\frac{x-\mu}{\sigma}\right)\right]^2}, \qquad x \in \mathbb{R}^1.$$

We remark that $g(x)$ is symmetric.

We point out that "Handbook of the logistic distribution" edited by Balakrishnan (1992) was published recently about the theory, the methodology and some applications of the family of logistic distributions, see also, Oliver(1964), Pearl and Reed (1920), Reed and Berkson(1929).

We denote $\boldsymbol{\theta} = (\mu, \sigma^2)^T$, and let $\hat{\boldsymbol{\theta}}_n = (\hat{\mu}_n, \hat{\sigma}_n^2)^T$ be the maximum of likelihood estimator of $\boldsymbol{\theta}$. Since there is no any other sufficient statistic for $\boldsymbol{\theta}$ than the trivial one $\mathbb{X} = (X_1, \ldots, X_n)^T$, the maximum likelihood equation has no explicit root. Balakrishnan and Cohen (1990) proposed an approximate solution of the maximum likelihood equations based on a "type II censured sample" of Harter and Moore (1967) (see also Grizzle (1961)). They proved that this approximate solution gives an asymptotically efficient estimator, i.e. asymptotically equivalent to $\hat{\boldsymbol{\theta}}_n$.

Let $\hat{\boldsymbol{\theta}}_n$ be such an estimator. As it follows from (21) the limit covariance matrix of the random vector $\sqrt{n}(\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta})$ will be $\mathbf{I}^{-1}$, where

$$(41)\qquad \mathbf{I} = \frac{1}{\sigma^2}\|I_{ij}\|_{2\times 2} = \frac{1}{9\sigma^2}\left\| \begin{matrix} \pi^2 & 0 \\ 0 & \pi^2+3 \end{matrix} \right\|,$$

$$I_{11} = \int_{-\infty}^{+\infty} \left[\frac{g'(x)}{g(x)}\right]^2 g(x)dx = \frac{\pi^2}{9}, \quad I_{22} = \int_{-\infty}^{+\infty} x^2 \left[\frac{g'(x)}{g(x)}\right]^2 g(x)dx - 1 = \frac{\pi^2+3}{9},$$

and since $g(x)$ is symmetric

$$I_{12} = I_{21} = \int_{-\infty}^{+\infty} x \left[\frac{g'(x)}{g(x)}\right]^2 g(x)dx = 0.$$

Let us fix the vector $\mathbf{p} = (p_1, p_2, \ldots, p_k)^T$ of positive probabilities such that

$$(42)\qquad p_1 = \ldots = p_k = 1/k,$$

and let

$$y_i = G^{-1}(p_1 + \cdots + p_i) = \frac{\sqrt{3}}{\pi}\ln\left(\frac{i}{k-i}\right), \quad i = 1, \ldots, k-1, \quad y_0 = -\infty, \quad y_k = +\infty.$$
(43)

Further, let $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_k)^T$ be the frequency vector arising from grouping $X_1, \ldots, X_n$ over the intervals with random ends

(44) $(-\infty, z_1], (z_1, z_2], \ldots, (z_{k-1}, +\infty)$, where $z_i = z_i(\hat{\boldsymbol{\theta}}_n) = \hat{\mu}_n + \hat{\sigma}_n y_i$,

and let

$$\mathbf{a} = (a_1, \ldots, a_k)^T, \quad \mathbf{b} = (b_1, \ldots, b_k)^T, \quad \mathbf{W}^T = -\frac{1}{\sigma} \parallel \mathbf{a}\vdots\mathbf{b} \parallel, \tag{45}$$

where for $i = 1, 2, \ldots, k$

$$\begin{aligned} a_i &= g(y_i) - g(y_{i-1}) = \frac{\pi}{k^2\sqrt{3}}(k - 2i + 1), \\ b_i &= y_i g(y_i) - y_{i-1} g(y_{i-1}) = \\ &= \frac{1}{k^2}\left[(i-1)(k-i+1)\ln\frac{k-i+1}{i-1} - i(k-i)\ln\frac{k-i}{i}\right], \end{aligned}$$

$$\alpha(\boldsymbol{\nu}) = k\sum_{i=1}^{k} a_i\nu_i = \frac{\pi}{\sqrt{3}k}\left[(k+1)n - 2\sum_{i=1}^{k} i\nu_i\right], \tag{46}$$

$$\beta(\boldsymbol{\nu}) = k\sum_{i=1}^{k} b_i\nu_i = \frac{1}{k}\sum_{i=1}^{k-1}(\nu_{i+1} - \nu_i)i(k-i)\ln\frac{k-i}{i}, \tag{47}$$

$$\lambda_1 = I_{11} - k\sum_{i=1}^{k} a_i^2 = \frac{\pi^2}{9k^2}, \qquad \lambda_2 = I_{22} - k\sum_{i=1}^{k} b_i^2. \tag{48}$$

Since g is symmetric we have $a_1 + a_2 + \cdots + a_k = b_1 + b_2 + \cdots + b_k = 0$. Let

$$\mathbf{B} = \mathbf{D} - \mathbf{p}^T\mathbf{p} - \mathbf{W}^T\mathbf{I}^{-1}\mathbf{W}, \tag{49}$$

where $\mathbf{D}$ is the diagonal matrix with the elements $1/k$ on the main diagonal. The matrix $\mathbf{B}$ does not depend on $\boldsymbol{\theta}$, and rank$\mathbf{B} = k - 1$, i.e. the matrix $\mathbf{B}$ is singular, while the matrix $\tilde{\mathbf{B}}$, obtained as a result of deleting the last row and column in $\mathbf{B}$, has an inverse

$$\tilde{\mathbf{B}}^{-1} = \mathbf{A} + \mathbf{A}\tilde{\mathbf{W}}^T(\mathbf{I} - \tilde{\mathbf{W}}\mathbf{A}\tilde{\mathbf{W}}^T)^{-1}\tilde{\mathbf{W}}\mathbf{A}, \tag{50}$$

where $\mathbf{A} = \tilde{\mathbf{D}}^{-1} + \mathbf{1}\mathbf{1}^T/p_k$, $\tilde{\mathbf{D}}^{-1}$ is a diagonal matrix with elements $\frac{1}{p_1}, \ldots, \frac{1}{p_{k-1}}$ on the main diagonal, $\mathbf{1} = \mathbf{1}_{k-1}$ is the vector of dimension $(k-1)$, all elements of which are equal to 1, $\tilde{\mathbf{W}}$ is a matrix obtained from $\mathbf{W}$ by deleting the last column. Since the vector $\tilde{\boldsymbol{\nu}} = (\nu_1, \ldots, \nu_{k-1})^T$ is asymptotically normally distributed with parameters

(51) $\mathbf{E}\tilde{\boldsymbol{\nu}} = n\tilde{\mathbf{p}} + O(\mathbf{1}_s)$ and $\mathbf{E}(\tilde{\boldsymbol{\nu}} - n\tilde{\mathbf{p}})^T(\tilde{\boldsymbol{\nu}} - n\tilde{\mathbf{p}}) = n\tilde{\mathbf{B}} + O(\mathbf{1}_{s\times s})$,

where $\tilde{\mathbf{p}} = (p_1, \ldots, p_{k-1})^T$, we obtain the next result

Theorem 1.

The statistic

$$Y_n^2 = \frac{1}{n}(\tilde{\boldsymbol{\nu}} - n\tilde{\mathbf{p}})^T\tilde{\mathbf{B}}^{-1}(\tilde{\boldsymbol{\nu}} - n\tilde{\mathbf{p}}) = X_n^2 + \frac{\lambda_1\beta^2(\boldsymbol{\nu}) + \lambda_2\alpha^2(\boldsymbol{\nu})}{n\lambda_1\lambda_2} \tag{52}$$

has, as $n \to \infty$, chi-squared limit distribution with $(k-1)$ degrees of freedom, where

$$X_n^2 = \sum_{i=1}^{k} \frac{(\nu_i - np_i)^2}{np_i} = \frac{k}{n}\sum_{i=1}^{k} \nu_i^2 - n. \tag{53}$$

Remark 5. We consider the hypothesis H_η according to which X_i follows $G(\frac{x-\mu}{\sigma}, \eta)$, where $G(x, \eta)$ is continuous, $|\, x \,| < \infty$, $\eta \in \mathbf{H} \subset \mathbb{R}^1$, $G(x, 0) = G(x)$, and $\eta = 0$ is a limit point of $\mathbf{H}$. Let us assume also, that

$$\frac{\partial}{\partial x}G(x, \eta) = g(x, \eta) \quad \text{and} \quad \frac{\partial}{\partial \eta}g(x, \eta)\mid_{\eta=0} = \Psi(x), \tag{54}$$

exist, where $g(x, 0) = g(x) = G'(x)$. In this case if $\frac{\partial^2 g(x,\eta)}{\partial \eta^2}$ exists and is continuous for all x in the neighbourhood of the $\eta = 0$, then

$$\mathbf{P}\{z_{i-1} < X_i \le z_i \mid H_\eta\} = p_i + \eta c_i + o(\eta), \tag{55}$$

where

$$c_i = \int_{z_{i-1}}^{z_i} \Psi(x)dx, \qquad i = 1, \ldots, k, \tag{56}$$

and finally, in the limit as $n \to \infty$ the statistic Y_n^2 has noncentral chi-squared distribution with $(k-1)$ degrees of freedom and with non-centrality parameter λ:

$$\lim_{n\to\infty} \mathbf{P}\{Y_n^2 \ge x \mid H_\eta\} = \mathbf{P}\{\chi_{k-1}^2(\lambda) \ge x\}, \tag{57}$$

where

$$\lambda = \sum_{i=1}^{k} \frac{c_i^2}{p_i} + \frac{\lambda_2\alpha^2(\mathbf{c}) + \lambda_1\beta^2(\mathbf{c})}{\lambda_1\lambda_2}, \mathbf{c} = (c_1, c_2, \ldots, c_k)^T,$$

$\mathbf{p}$, $\alpha(\mathbf{c})$, $\beta(\mathbf{c})$, λ_1, λ_2 are given by (42),(46),(47),(48) respectively.

6. EXAMPLE (ABOUT A CHOICE OF INTERVALS)

1. Simple hypotheses. Suppose that we want to test the simple hypothesis H_0 acording to which

$$\mathbf{P}\{X_1 \leq x \mid H_0\} = G(x) \tag{1}$$

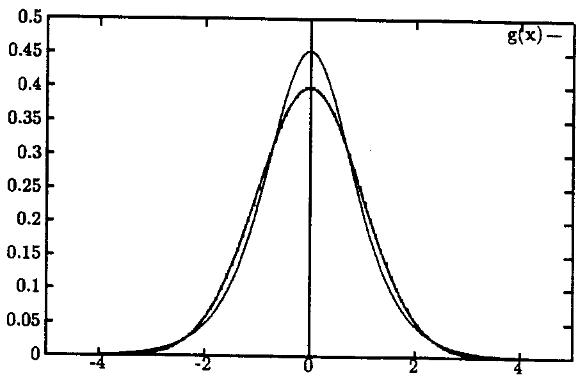
against the simple hypothesis H_1:

$$\mathbf{P}\{X_1 \leq x \mid H_1\} = \Phi(x) \tag{2}$$

($\Phi(x)$ being the standard normal distribution function).

It would be possible to use the Neymann and Pearson test, yet it would entrail large calculations. We shall then try to adapt a chi-squared test.

Let $(X_1, ..., X_n)^T$ be a sample of mutually independent identically distributed random variables with $\mathbf{E}(X_1) = 0$, $\mathbf{Var}(X_1) = 1$. Before to construct a chi-square test for testing H_0 against H_1 we shall do one remark on the cells' choice. As the test only compares the respective frequencies on the cells, it is worthwhile to choose those when both density curves are the most distant, that is to say those got by their junctions.



The representative curves of the density function $g(x)$ of $L(0,1)$ distribution and the density function $\varphi(x)$ of the standard normal $N(0,1)$ distribution have four symmetric points of intersection:

$$x_1 = -x_6 = -\infty, \qquad x_2 = -x_5, \qquad x_3 = -x_4,$$

where $\varphi(x_i) = g(x_i)$. Let

$$I_i = \{x_{i-1} < x \leq x_i\}, \qquad i = 1, 2, 3, 4, 5,$$

be the such definite intervals of grouping the data. We can even improve the power of the test by considering the cells only differentiated by the relative positions of the two curves:

$$\begin{aligned} J_1 &= I_1 \cup I_3 \cup I_5 = \{x : g(x) > \varphi(x)\}, \quad g(x) \text{ higher that } \varphi(x), \\ J_2 &= I_2 \cup I_4 = \{x : \varphi(x) > g(x)\}, \quad \varphi(x) \text{ higher that } g(x). \end{aligned}$$

Let us remark that if we consider, for example, an interval $-2 < x \leq 0$ for grouping the data with one point of intersection x_3 on the inside, as shown on the schema, it is clear that in this case the two probabilities

$$\mathbf{P}\{-2 < X_i \leq 0 \mid H_0\} \text{ and } \mathbf{P}\{-2 < X_i \leq 0 \mid H_1\}$$

will be approximately equal, and it will be difficult to decide which of both hypotheses is true and hence the power of the test using such interval will be low. A test using the intervals $\{-2 < x \leq x_3\}$ and $\{x_3 < x \leq 0\}$ (or $x_3 < x \leq x_4\}$) will obviously be more powerfull. From this remark we obtain the next

Proposition

To test H_0 against H_1, the chi-squared test using I_1, I_2, I_3, I_4, I_5 is less powerful as the one using two cells:

$$J_1 \text{ and } J_2.$$

a) Let $(\nu_1, \nu_2, \ldots, \nu_5)^T$ be the observed frequencies of the sample in the intervals $I_1, \ldots, I_5$ respectively. Then

$$\mathbf{P}\{X_1 \in I_i \mid H_0\} = p_i^{(0)}, \qquad i = 1, 2, \ldots, 5 \tag{60}$$

(with $p_1^{(0)} = p_5^{(0)} = 0.0155$; $p_2^{(0)} = p_4^{(0)} = 0.207$; $p_3^{(0)} = 0.555$). In our case we shall use the standard statistic of Pearson:

$$X_n^2 = \sum_{i=1}^{5} \frac{(\nu_i - np_i^{(0)})^2}{np_i^{(0)}}. \tag{61}$$

Under H_0, X_n^2 is asymptotically χ_4^2 and we shall reject H_0 if $X_n^2 \geq c_{4,\alpha}$, since

$$\mathbf{P}\{X_n^2 \geq c_{4,\alpha} \mid H_0\} \approx \mathbf{P}\{\chi_4^2 \geq c_{4,\alpha}\} = \alpha.$$

The power of this test is $\mathcal{P}_5 = \mathbf{P}\{X_n^2 \geq c_{4,\alpha} \mid H_1\}$. Or,

$$\mathbf{P}\{X_1 \in I_i \mid H_1\} = p_i^{(1)}, \qquad i = 1, 2, \ldots, 5, \tag{62}$$

$(p_1^{(1)} = p_5^{(1)} = 0.011; p_2^{(1)} = p_4^{(1)} = 0.234; p_3^{(1)} = 0.51)$.

Under H_1, X_n^2 has asymptotically a non-central chi-square distribution with 4 degrees of freedom and the parameter of non centrality λ:

$$\lambda = n \sum_{i=1}^{5} \frac{(p_i^{(0)} - p_i^{(1)})^2}{p_i^{(0)}}, \qquad (\lambda \sim 0.0133n).$$

Using the approximation of Patnaik we can compute

(63) $$\mathbf{P}\{\chi_{k-1}^2(\lambda) \geq c_\alpha\} \sim \mathbf{P}\{\chi_{k-1}^2 \geq c_\alpha \left(1 - \frac{\lambda}{k-1}\right)\}.$$

For example, for $\alpha = 0.05$ we have $c_{4,\alpha} = 9.49$, from which it follows that

in order to have $\mathcal{P}_5$	>	0.5	0.74	0.9	0.99,
it is necessary to take n	>	195	240	269	292

respectively.

b) Let ν be the observed frequency on J_1. We have

$$\mathbf{P}\{X_1 \in J_1 \mid H_0\} = \omega^{(0)} \text{ with } \omega^{(0)} = 0.586$$

and

(64) $$\mathbf{P}\{X_1 \in J_1 \mid H_1\} = \omega^{(1)} \text{ with } \omega^{(1)} = 0.532.$$

To test H_0 against H_1 we shall use again the standard statistic of Pearson

$$X_n^2 = \frac{(\nu - n\omega^{(0)})^2}{n\omega^{(0)}(1 - \omega^{(0)})}.$$

Under H_0 the statistic X_n^2 is distributed ($n \to \infty$) asymptotically as χ_1^2. We shall reject H_0 if $X_n^2 \geq c_{1,\alpha}$, since $\mathbf{P}\{\chi_1^2 \geq c_{1,\alpha}\} = \alpha$ and the power of this test is

$$\mathcal{P}_2 = \mathbf{P}\{X_n^2 \geq c_{1,\alpha} \mid H_1\}.$$

Under H_1, X_n^2 has asymptotically ($n \to \infty$) a non central chi-square distribution with one degree of freedom and the parameter of noncentrality

$$\lambda' = \frac{[\omega^{(0)} - \omega^{(1)}]^2}{\omega^{(0)}(1 - \omega^{(0)})}, \qquad (\lambda' \sim 0.012n).$$

Using the same approximation (63):

$$\mathbf{P}\{\chi_1^2(\lambda') \geq c_\alpha\} \sim \mathbf{P}\{\chi_1^2 \geq c_{1,\alpha}(1 - \lambda')\},$$

we have $c_{1,\alpha} = 3.84$ for $\alpha = 0.05$, from which it follows that

$$\begin{array}{lccccc} \text{in order to have } \mathcal{P}_2 & > & 0.5 & 0.75 & 0.9 & 0.99, \\ \text{it is necessary to take } n & > & 74 & 82 & 83 & 84 \end{array}$$

respectively. One can see that in the case b) the chi-square test based on J_1 and J_2 is more powerfull, then in the case a). The same approach may be used for composite hypotheses (see, for example, Nikulin & Voinov, 1989).

2. Composite hypotheses. Let now $\mathbb{X} = (X_1, \ldots, X_n)^T$ be a sample, $\mathbf{E}(X_1) = \mu$ and $\mathbf{Var}(X_1) = \sigma^2$, $\boldsymbol{\theta} = (\mu, \sigma^2)^T$, $\boldsymbol{\theta}$ is unknown, and let us test H_0 according to which X_1 follows a logistic distribution (39):

$$\mathbf{P}\{X_1 \leq x \mid H_0\} = G(x, \boldsymbol{\theta}) = G(\frac{x-\mu}{\sigma}) \tag{65}$$

against the hypothesis of normality H_1 according to which

$$\mathbf{P}\{X_1 \leq x \mid H_1\} = \Phi(\frac{x-\mu}{\sigma}). \tag{66}$$

Let $\hat{\boldsymbol{\theta}}_n$ be an estimator which satisfies to (21). According to the precedent study and to §5, we shall take the two cells with random boundaries (41):

$$\begin{aligned} J_1(\hat{\boldsymbol{\theta}}_n) &=]-\infty, -\hat{\sigma}_n x_5 + \hat{\mu}_n] \cup] - \hat{\sigma}_n x_3 + \hat{\mu}_n, \hat{\sigma}_n x_3 + \hat{\mu}_n] \cup] \hat{\sigma}_n x_5 + \hat{\mu}_n, +\infty[, \\ J_2(\hat{\boldsymbol{\theta}}_n) &= \mathbb{R}^1 \setminus J_1(\hat{\boldsymbol{\theta}}_n). \end{aligned}$$

Let $\boldsymbol{\nu} = (\nu, n-\nu)^T$ be the vector of frequencies obtained in the result of the groupement of the sample $\mathbb{X} = (X_1, \ldots, X_n)^T$ into the intervals $J_1(\hat{\boldsymbol{\theta}}_n)$ and $J_2(\hat{\boldsymbol{\theta}}_n)$. According to definitions (43) to (47) we calculate

$$\begin{aligned} a_1' &= -\sigma \frac{\partial}{\partial \mu} \mathbf{P}\{X_i \in J_1(\boldsymbol{\theta}) \mid H_0\} = a_1 + a_3 + a_5 = 0 = a_2', \\ b_1' &= -\sigma \frac{\partial}{\partial \sigma} \mathbf{P}\{X_i \in J_1(\boldsymbol{\theta}) \mid H_0\} = \\ &= b_1 + b_3 + b_5 = 2[x_3 g(x_3) - x_5 g(x_5)] = -b_2' \approx 0.296, \\ \lambda_1 &= I_{11} - \frac{a_1'^2}{\omega^{(0)}} - \frac{a_2'^2}{1-\omega^{(0)}} = I_{11} = \frac{\pi^2}{9}, \end{aligned}$$

$$
\begin{aligned}
\lambda_2 &= I_{22} - \frac{b_1'^2}{\omega^{(0)}} - \frac{b_2'^2}{1-\omega^{(0)}} = \frac{\pi^2+3}{9} - \frac{b_1'^2}{\omega^{(0)}(1-\omega^{(0)})}, \\
\lambda_3 &= I_{12} - 0 = 0, \\
\alpha(\boldsymbol{\nu}) &= \frac{a_1'\nu}{\omega^{(0)}} + \frac{a_2'(n-\nu)}{1-\omega^{(0)}} = 0, \\
\beta(\boldsymbol{\nu}) &= \frac{b_1'\nu}{\omega^{(0)}} + \frac{b_2'(n-\nu)}{1-\omega^{(0)}} = b_1' \frac{(\nu - n\omega^{(0)})}{\omega^{(0)}(1-\omega^{(0)})}, \\
X_n^2 &= \frac{(\nu - n\omega^{(0)})^2}{n\omega^{(0)}(1-\omega^{(0)})},
\end{aligned}
$$

and finally we obtain the statistic

$$
(67) \qquad Y_n^2 = X_n^2 + \frac{\beta(\boldsymbol{\nu})}{n\lambda_2} = X_n^2 + \frac{\frac{b_1'^2}{\omega^{(0)}(1-\omega^{(0)})}}{\left[\frac{\pi^2+3}{9} - \frac{b_1'^2}{\omega^{(0)}(1-\omega^{(0)})}\right]} X_n^2 \sim 1.34 X_n^2,
$$

for testing the composite hypothesis H_0, given by (39), against the hypothesis of normality H_1, and

$$
\mathbf{P}\{Y_n^2 \geq x \mid H_0\} \to \mathbf{P}\{\chi_1^2 \geq x\}, \qquad (n \to \infty).
$$

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