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Muliere and Scarsini's bivariate Pareto distribution: sums, products, and ratios

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Abstract

We derive the exact distributions of $R = X + Y$, $P = XY$ and $W = X/(X + Y)$ and the corresponding moment properties when X and Y follow Muliere and Scarsini's bivariate Pareto distribution. The expressions turn out to involve special functions. We also provide extensive tabulations of the percentage points associated with the distributions. These tables –obtained using intensive computing power– will be of use to practitioners of the bivariate Pareto distribution.

MSC: 33C90, 62E99

Keywords: incomplete beta function, Gauss hypergeometric function, Muliere and Scarsini's bivariate Pareto distribution, products of random variables, ratios of random variables, sums of random variables.

1 Introduction

Since the 1930s, the statistics literature has seen many developments in the theory and applications of linear combinations and ratios of random variables. Some of these include:

- Ratios of normal random variables appear as sampling distributions in single equation models, in simultaneous equations models, as posterior distributions for parameters of regression models and as modeling distributions, especially in

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economics when demand models involve the indirect utility function (details in Yatchew, 1986).

- Weighted sums of uniform random variables –in addition to the well known application to the generation of random variables– have applications in stochastic processes which in many cases can be modeled by these weighted sums. In computer vision algorithms these weighted sums play a pivotal role (Kamgar-Parsi *et al.*, 1995). An earlier application of the linear combinations of uniform random variables is given in connection with the distribution of errors in nth tabular differences Δ^n (Lowan and Laderman, 1939).
- Ratio of linear combinations of chi-squared random variables are part of von Neumann's (1941) test statistics (mean square successive difference divided by the variance). These ratios appear in various two-stage tests (Toyoda and Ohtani, 1986). They are also used in tests on structural coefficients of a multivariate linear functional relationship model (details in Chaubey and Nur Enayet Talukder (1983) and Provost and Rudiuk (1994)).
- Sums of independent gamma random variables have applications in queuing theory problems such as determination of the total waiting time and in civil engineering problems such as determination of the total excess water flow into a dam. They also appear in test statistics used to determine the confidence limits for the coefficient of variation of fiber diameters (Linhart (1965) and Jackson (1969)) and in connection with the inference about the mean of the two-parameter gamma distribution (Grice and Bain, 1980).
- Linear combinations of inverted gamma random variables are used for testing hypotheses and interval estimation based on generalized p-values, specifically for the Behrens-Fisher problem and variance components in balanced mixed linear models (Witkovský, 2001).
- As to the Beta distributions their linear combinations occur in calculations of the power of a number of tests in ANOVA (Monti and Sen, 1976) among other applications. More generally, the linear combinations are used for detecting changes in the location of the distribution of a sequence of observations in quality control problems (Lai, 1974). Pham-Gia and Turkkan (1993, 1994, 1998, 2002) and Pham-Gia (2000) provided applications of sums and ratios to availability, Bayesian quality control and reliability.
- Linear combinations of the form $T = a_1 t_{f_1} + a_2 t_{f_2}$, where t_f denotes the Student t random variable based on f degrees of freedom, represents the Behrens-Fisher statistic and – as early as the middle of the twentieth century – Stein (1945) and Chapman (1950) developed a two-stage sampling procedure involving the T to test whether the ratio of two normal random variables is equal to a specified constant.

- Weighted sums of the Poisson parameters are used in medical applications for directly standardized mortality rates (Dobson *et al.*, 1991).

In this paper, we consider the distributions of $R = X+Y$, $P = XY$ and $W = X/(X+Y)$ when X and Y are correlated Pareto random variables with the joint survivor function expressed as the mixture of two components:

$$\bar{F}(x, y) = \frac{\lambda_1 + \lambda_2}{\lambda_0 + \lambda_1 + \lambda_2} \bar{F}_a(x, y) + \frac{\lambda_0}{\lambda_0 + \lambda_1 + \lambda_2} \bar{F}_s(x, y), \quad (1)$$

where $\bar{F}_a(x, y)$ is the absolutely continuous part with respect to Lebesgue measure given by

$$\bar{F}_a(x, y) = \left(\frac{x}{\beta}\right)^{-\lambda_1} \left(\frac{y}{\beta}\right)^{-\lambda_2} \left\{\max\left(\frac{x}{\beta}, \frac{y}{\beta}\right)\right\}^{-\lambda_0} \quad (2)$$

and $\bar{F}_s(x, y)$ is the singular part concentrating on the line $x = y$ given by

$$\bar{F}_s(x, y) = \left\{\max\left(\frac{x}{\beta}, \frac{y}{\beta}\right)\right\}^{-(\lambda_0+\lambda_1+\lambda_2)} \quad (3)$$

for $x \geq \beta$, $y \geq \beta$, $\beta > 0$, $\lambda_0 > 0$, $\lambda_1 > 0$ and $\lambda_2 > 0$. The joint density of the absolutely continuous part is:

$$f_a(x, y) = \begin{cases} \dfrac{\lambda_2(\lambda_0 + \lambda_1)}{\beta^2} \left(\dfrac{x}{\beta}\right)^{-(1+\lambda_0+\lambda_1)} \left(\dfrac{y}{\beta}\right)^{-(1+\lambda_2)}, & \text{if } x > y \geq \beta, \\[2ex] \dfrac{\lambda_1(\lambda_0 + \lambda_2)}{\beta^2} \left(\dfrac{y}{\beta}\right)^{-(1+\lambda_0+\lambda_2)} \left(\dfrac{x}{\beta}\right)^{-(1+\lambda_1)}, & \text{if } y > x \geq \beta. \end{cases}$$

This distribution is due to Muliere and Scarsini (1987) and therefore known as Muliere and Scarsini's bivariate Pareto distribution. It has received applications in several areas especially in reliability (see, for example, Kotz *et al* (2000)).

The paper is organized as follows. In Sections 2 and 3, we derive explicit expressions for the pdfs and moments of $R = X + Y$, $P = XY$ and $W = X/(X + Y)$. In Section 4, we provide extensive tabulations of the associated percentage points, obtained by means of intensive computing power. These values will be of use to the practitioners of the bivariate Pareto distribution.

The calculations of this paper involve several special functions, including the incomplete beta function defined by

$$B_x(a, b) = \int_0^x t^{a-1}(1-t)^{b-1} dt$$

and, the Gauss hypergeometric function defined by

$$G(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!},$$

where $(e)_k = e(e+1)\cdots(e+k-1)$ denotes the ascending factorial. We also need the following important lemma.

Lemma 1 *(Equation (3.194.2), Gradshteyn and Ryzhik, 2000) For $\mu > \nu$,*

$$\int_u^{\infty} \frac{x^{\mu-1}}{(1+\beta x)^{\nu}} dx = \frac{u^{\mu-\nu}}{\beta^{\nu}(\nu-\mu)} G\left(\nu, \nu-\mu; \nu-\mu+1; -\frac{1}{\beta u}\right).$$

The properties of the above special functions can be found in Prudnikov *et al.* (1986) and Gradshteyn and Ryzhik (2000).

2 Probability density functions

Theorems 1 to 3 derive the pdfs of $R = X + Y$, $P = XY$ and $W = X/(X+Y)$ when X and Y are distributed according to (1)–(3).

Theorem 1 *If X and Y are jointly distributed according to (1)–(3) then*

$$f_R(r) = \frac{\lambda_0 (2\beta)^{\lambda_0+\lambda_1+\lambda_2}}{r^{1+\lambda_0+\lambda_1+\lambda_2}} + \frac{(\lambda_0+\lambda_1)(\lambda_1+\lambda_2)\beta^{\lambda_0+\lambda_1+\lambda_2}}{(\lambda_0+\lambda_1+\lambda_2) r^{1+\lambda_0+\lambda_1+\lambda_2}} K_1(r) + \frac{(\lambda_0+\lambda_2)(\lambda_1+\lambda_2)\beta^{\lambda_0+\lambda_1+\lambda_2}}{(\lambda_0+\lambda_1+\lambda_2) r^{1+\lambda_0+\lambda_1+\lambda_2}} K_2(r) \quad (4)$$

for $2\beta \leq r < \infty$, where

$$K_1(r) = \left(\frac{r}{\beta} - 1\right)^{\lambda_2} G\left(-\lambda_0-\lambda_1-\lambda_2, -\lambda_2; 1-\lambda_2; -\frac{\beta}{r-\beta}\right) - G(-\lambda_0-\lambda_1-\lambda_2, -\lambda_2; 1-\lambda_2; -1)$$

and

$$K_2(r) = G(-\lambda_0-\lambda_1-\lambda_2, -\lambda_1; 1-\lambda_1; -1) - \left(\frac{r}{\beta} - 1\right)^{\lambda_1} G\left(-\lambda_0-\lambda_1-\lambda_2, -\lambda_1; 1-\lambda_1; -\frac{\beta}{r-\beta}\right).$$

Proof. Set $(R, W) = (X + Y, X/R)$ and note that the Jacobian is R for the continuous part and $1/2$ for the singular part. From (1)–(3), the joint pdf of (R, W) can be written as

$$f(r, w) = \frac{\lambda_1 + \lambda_2}{\lambda_0 + \lambda_1 + \lambda_2} f_a(r, w) + \frac{\lambda_0}{\lambda_0 + \lambda_1 + \lambda_2} f_s(r, w), \tag{5}$$

where

$$f_a(r, w) = \begin{cases} \lambda_2 (\lambda_0 + \lambda_1) \beta^{\lambda_0+\lambda_1+\lambda_2} r (rw)^{-(1+\lambda_0+\lambda_1)} (r(1-w))^{-(1+\lambda_2)}, & \text{if } w > 1/2, \\ \lambda_1 (\lambda_0 + \lambda_2) \beta^{\lambda_0+\lambda_1+\lambda_2} r (rw)^{-(1+\lambda_1)} (r(1-w))^{-(1+\lambda_0+\lambda_2)}, & \text{if } w < 1/2 \end{cases} \tag{6}$$

and

$$f_s(r, w) = (1/2)(\lambda_0 + \lambda_1 + \lambda_2)\beta^{\lambda_0+\lambda_1+\lambda_2} (rw)^{-(1+\lambda_0+\lambda_1+\lambda_2)}. \tag{7}$$

Note that $f_a(r, w)$ is the joint density of the absolutely continuous part and that $f_s(r, w)$ is the density of the singular part along the line $w = 1/2$. Thus, the pdf of R can be written as

$$\begin{aligned} f_R(r) &= \frac{\lambda_0 (2\beta)^{\lambda_0+\lambda_1+\lambda_2}}{r^{1+\lambda_0+\lambda_1+\lambda_2}} + \frac{\lambda_2 (\lambda_0 + \lambda_1)(\lambda_1 + \lambda_2)\beta^{\lambda_0+\lambda_1+\lambda_2}}{(\lambda_0 + \lambda_1 + \lambda_2) r^{1+\lambda_0+\lambda_1+\lambda_2}} I_1(r) \\ &\quad + \frac{\lambda_1 (\lambda_0 + \lambda_2)(\lambda_1 + \lambda_2)\beta^{\lambda_0+\lambda_1+\lambda_2}}{(\lambda_0 + \lambda_1 + \lambda_2) r^{1+\lambda_0+\lambda_1+\lambda_2}} I_2(r), \end{aligned} \tag{8}$$

where

$$I_1(r) = \int_{1/2}^{1-\beta/r} w^{-(1+\lambda_0+\lambda_1)} (1-w)^{-(1+\lambda_2)} dw$$

and

$$I_2(r) = \int_{\beta/r}^{1/2} w^{-(1+\lambda_1)} (1-w)^{-(1+\lambda_0+\lambda_2)} dw.$$

These integrals can be calculated by application of Lemma 1. Setting $u = w/(1-w)$, one can calculate

$$\begin{aligned} I_1(r) &= \int_1^{r/\beta-1} u^{-(1+\lambda_0+\lambda_1)} (1+u)^{\lambda_0+\lambda_1+\lambda_2} du \\ &= \lambda_2^{-1} K_1(r), \end{aligned} \tag{9}$$

where the second step follows by application of Lemma 1. Similarly, setting $u =$

$(1 - w)/w$, one can show that

$$I_2(r) = \int_{r/\beta-1}^{1} u^{-(1+\lambda_0+\lambda_2)}(1+u)^{\lambda_0+\lambda_1+\lambda_2} du = \lambda_1^{-1} K_2(r). \quad (10)$$

The result of the theorem follows by substituting (9) and (10) into (8). ■

Theorem 2 *If X and Y are jointly distributed according to (1)–(3) then*

$$\begin{aligned} f_P(p) = & \frac{\lambda_2(\lambda_0+\lambda_1)(\lambda_1+\lambda_2)}{(\lambda_2-\lambda_1-\lambda_0)(\lambda_0+\lambda_1+\lambda_2)} \beta^{\lambda_0+\lambda_1+\lambda_2} p^{-(\lambda_0+\lambda_1+\lambda_2)/2-1} \left\{ \beta^{\lambda_0+\lambda_1-\lambda_2} p^{(\lambda_2-\lambda_1-\lambda_0)/2} - 1 \right\} \\ & +(1/2)\lambda_0 \beta^{\lambda_0+\lambda_1+\lambda_2} p^{-(\lambda_0+\lambda_1+\lambda_2)/2-1} \\ & + \frac{\lambda_1(\lambda_0+\lambda_2)(\lambda_1+\lambda_2)}{(\lambda_0+\lambda_2-\lambda_1)(\lambda_0+\lambda_1+\lambda_2)} \beta^{\lambda_0+\lambda_1+\lambda_2} p^{-(1+\lambda_0+\lambda_2)} \left\{ p^{(\lambda_0+\lambda_2-\lambda_1)/2} - \beta^{\lambda_0+\lambda_2-\lambda_1} \right\} \end{aligned} \quad (11)$$

for $\beta^2 < p < \infty$.

Proof. Set $(X, P) = (X, XY)$ and note that the Jacobian is $1/X$. From (1)–(3), the joint pdf of (X, P) can be written as

$$f(x, p) = \frac{\lambda_1+\lambda_2}{\lambda_0+\lambda_1+\lambda_2} f_a(x, p) + \frac{\lambda_0}{\lambda_0+\lambda_1+\lambda_2} f_s(x, p),$$

where

$$f_a(x, p) = \begin{cases} \lambda_2(\lambda_0+\lambda_1)\beta^{\lambda_0+\lambda_1+\lambda_2} x^{-(2+\lambda_0+\lambda_1)} (p/x)^{-(1+\lambda_2)}, & \text{if } x > \sqrt{p}, \\ \lambda_1(\lambda_0+\lambda_2)\beta^{\lambda_0+\lambda_1+\lambda_2} x^{-(2+\lambda_1)} (p/x)^{-(1+\lambda_0+\lambda_2)}, & \text{if } x < \sqrt{p} \end{cases}$$

and

$$f_s(x, p) = (1/2)(\lambda_0+\lambda_1+\lambda_2)\beta^{\lambda_0+\lambda_1+\lambda_2} p^{-(\lambda_0+\lambda_1+\lambda_2)/2-1}.$$

Note that $f_a(x, p)$ is the joint density of the absolutely continuous part and that $f_s(x, p)$ is the density of the singular part along the line $x = \sqrt{p}$. Thus, the pdf of P can be written as

$$
\begin{aligned}
f_P(p) &= \frac{\lambda_2 (\lambda_0 + \lambda_1)(\lambda_1 + \lambda_2)}{\lambda_0 + \lambda_1 + \lambda_2} \beta^{\lambda_0+\lambda_1+\lambda_2} p^{-(1+\lambda_2)} \int_{\sqrt{p}}^{p/\beta} x^{-(1+\lambda_0+\lambda_1-\lambda_2)} dx \\
&\quad + (1/2)\lambda_0 \beta^{\lambda_0+\lambda_1+\lambda_2} p^{-(\lambda_0+\lambda_1+\lambda_2)/2-1} \\
&\quad + \frac{\lambda_1 (\lambda_0 + \lambda_2)(\lambda_1 + \lambda_2)}{\lambda_0 + \lambda_1 + \lambda_2} \beta^{\lambda_0+\lambda_1+\lambda_2} p^{-(1+\lambda_0+\lambda_2)} \int_{\beta}^{\sqrt{p}} x^{\lambda_0+\lambda_2-\lambda_1-1} dx.
\end{aligned}
$$

The result of the theorem follows by elementary integration of the above integrals. ∎

Theorem 3 *If X and Y are jointly distributed according to (1)–(3) then*

$$
f_W(w) = \begin{cases}
\dfrac{\lambda_2 (\lambda_0 + \lambda_1)(\lambda_1 + \lambda_2)}{(\lambda_0 + \lambda_1 + \lambda_2)^2} \dfrac{(1-w)^{\lambda_0+\lambda_1-1}}{w^{\lambda_0+\lambda_1+1}}, & \text{if } w > 1/2, \\[2ex]
\dfrac{\lambda_0}{\lambda_0 + \lambda_1 + \lambda_2}, & \text{if } w = 1/2, \\[2ex]
\dfrac{\lambda_1 (\lambda_0 + \lambda_2)(\lambda_1 + \lambda_2)}{(\lambda_0 + \lambda_1 + \lambda_2)^2} \dfrac{w^{\lambda_0+\lambda_2-1}}{(1-w)^{\lambda_0+\lambda_2+1}}, & \text{if } w < 1/2
\end{cases} \tag{12}
$$

for $0 < w < 1$.

Proof. Using (5)–(7), one can write

$$
f_W(w) = \begin{cases}
\dfrac{\lambda_2 (\lambda_0 + \lambda_1)(\lambda_1 + \lambda_2)}{\lambda_0 + \lambda_1 + \lambda_2} \beta^{\lambda_0+\lambda_1+\lambda_2} w^{-(1+\lambda_0+\lambda_1)} (1-w)^{-(1+\lambda_2)} \\
\qquad \times \displaystyle\int_{\beta/(1-w)}^{\infty} r^{-(1+\lambda_0+\lambda_1+\lambda_2)} dr \qquad \text{if } w > 1/2, \\[2ex]
(1/2)\lambda_0 \beta^{\lambda_0+\lambda_1+\lambda_2} w^{-(1+\lambda_0+\lambda_1+\lambda_2)} \displaystyle\int_{2\beta}^{\infty} r^{-(1+\lambda_0+\lambda_1+\lambda_2)} dr, \\
\qquad \qquad \text{if } w = 1/2, \\[2ex]
\dfrac{\lambda_1 (\lambda_0 + \lambda_2)(\lambda_1 + \lambda_2)}{\lambda_0 + \lambda_1 + \lambda_2} \beta^{\lambda_0+\lambda_1+\lambda_2} w^{-(1+\lambda_1)} (1-w)^{-(1+\lambda_0+\lambda_2)} \\
\qquad \times \displaystyle\int_{\beta/w}^{\infty} r^{-(1+\lambda_0+\lambda_1+\lambda_2)} dr \qquad \text{if } w < 1/2.
\end{cases} \tag{13}
$$

The result of the theorem follows by elementary integration of the above integrals. ∎

Using special properties of the hypergeometric functions, one can derive elementary forms for the pdf in (4). This is illustrated in the corollaries below.

Corollary 1 *If X and Y are jointly distributed according to (1)–(3) and if* $\lambda_1 \geq 1$ *is an integer then the pdf of R is given by (4) with*

$$K_2(r) = \sum_{k=0}^{\lambda_1} \frac{(-\lambda_0 - \lambda_1 - \lambda_2)_k (-\lambda_1)_k (-1)^k}{(1-\lambda_1)_k \, k!} - \left(\frac{r}{\beta} - 1\right)^{\lambda_1} \sum_{k=0}^{\lambda_1} \frac{(-\lambda_0 - \lambda_1 - \lambda_2)_k (-\lambda_1)_k (-1)^k}{(1-\lambda_1)_k \, k!} \left(\frac{\beta}{r-\beta}\right)^k.$$

Corollary 2 *If X and Y are jointly distributed according to (1)–(3) and if* $\lambda_2 \geq 1$ *is an integer then the pdf of R is given by (4) with*

$$K_1(r) = \left(\frac{r}{\beta} - 1\right)^{\lambda_2} \sum_{k=0}^{\lambda_2} \frac{(-\lambda_0 - \lambda_1 - \lambda_2)_k (-\lambda_2)_k (-1)^k}{(1-\lambda_2)_k \, k!} \left(\frac{\beta}{r-\beta}\right)^k - \sum_{k=0}^{\lambda_2} \frac{(-\lambda_0 - \lambda_1 - \lambda_2)_k (-\lambda_2)_k (-1)^k}{(1-\lambda_2)_k \, k!}.$$

Corollary 3 *If X and Y are jointly distributed according to (1)–(3) and if* $\lambda_0+\lambda_1+\lambda_2 \geq 1$ *is an integer then the pdf of R is given by (4) with*

$$K_1(r) = \left(\frac{r}{\beta} - 1\right)^{\lambda_2} \sum_{k=0}^{\lambda_0+\lambda_1+\lambda_2} \frac{(-\lambda_0 - \lambda_1 - \lambda_2)_k (-\lambda_2)_k (-1)^k}{(1-\lambda_2)_k \, k!} \left(\frac{\beta}{r-\beta}\right)^k - \sum_{k=0}^{\lambda_0+\lambda_1+\lambda_2} \frac{(-\lambda_0 - \lambda_1 - \lambda_2)_k (-\lambda_2)_k (-1)^k}{(1-\lambda_2)_k \, k!}$$

and

$$K_2(r) = \sum_{k=0}^{\lambda_0+\lambda_1+\lambda_2} \frac{(-\lambda_0 - \lambda_1 - \lambda_2)_k (-\lambda_1)_k (-1)^k}{(1-\lambda_1)_k \, k!} - \left(\frac{r}{\beta} - 1\right)^{\lambda_1} \sum_{k=0}^{\lambda_0+\lambda_1+\lambda_2} \frac{(-\lambda_0 - \lambda_1 - \lambda_2)_k (-\lambda_1)_k (-1)^k}{(1-\lambda_1)_k \, k!} \left(\frac{\beta}{r-\beta}\right)^k.$$

Figures 1 to 3 illustrate the shape of the pdfs (4), (11) and (12) for selected values of λ_0, λ_1 and λ_2. Each plot contains four curves corresponding to selected values of λ_1 and λ_2. The effect of the parameters is evident.

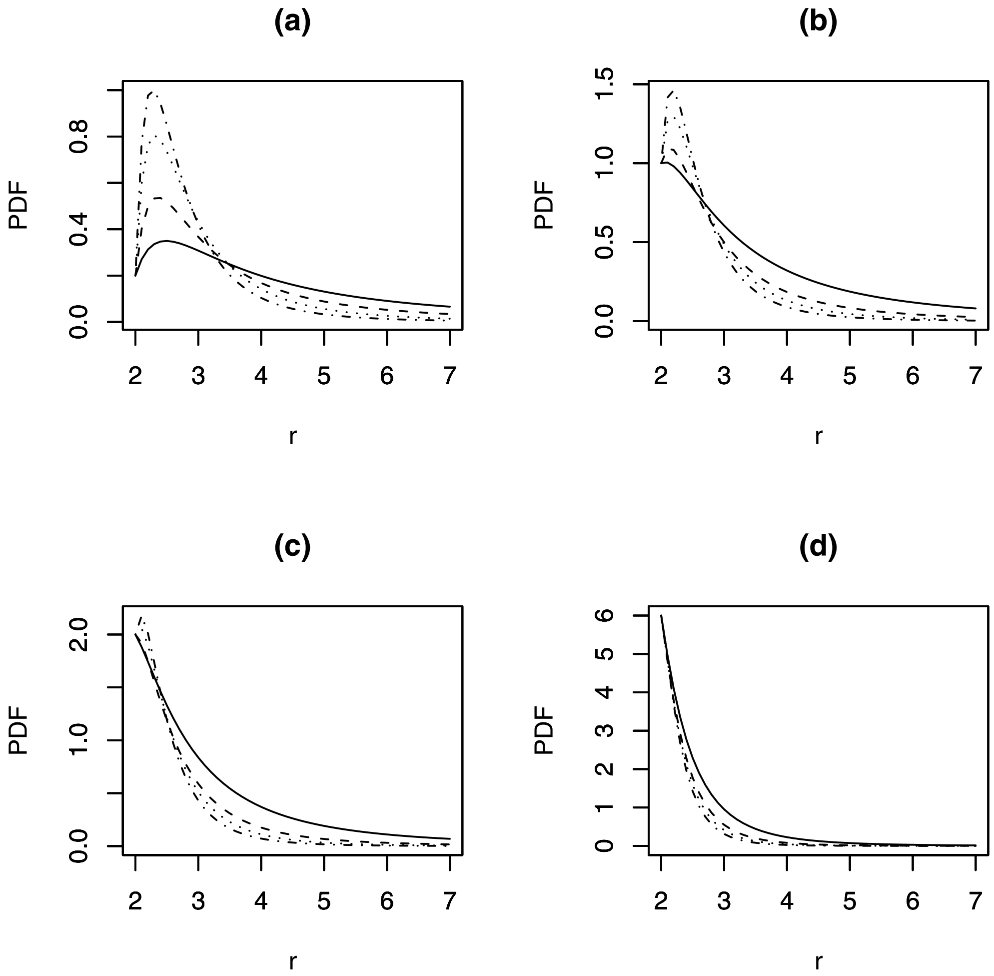

Figure 1: *Plots of the pdf of (4) for* $\beta = 1$ *and (a):* $\lambda_0 = 0.1$*; (b):* $\lambda_0 = 0.5$*; (c):* $\lambda_0 = 1$*; and, (d):* $\lambda_0 = 3$*. The four curves in each plot are: the solid curve (*$\lambda_1 = 1$, $\lambda_2 = 1$*), the curve of lines (*$\lambda_1 = 3$, $\lambda_2 = 1$*), the curve of dots (*$\lambda_1 = 3$, $\lambda_2 = 2$*), and the curve of lines and dots (*$\lambda_1 = 3$, $\lambda_2 = 3$*).*

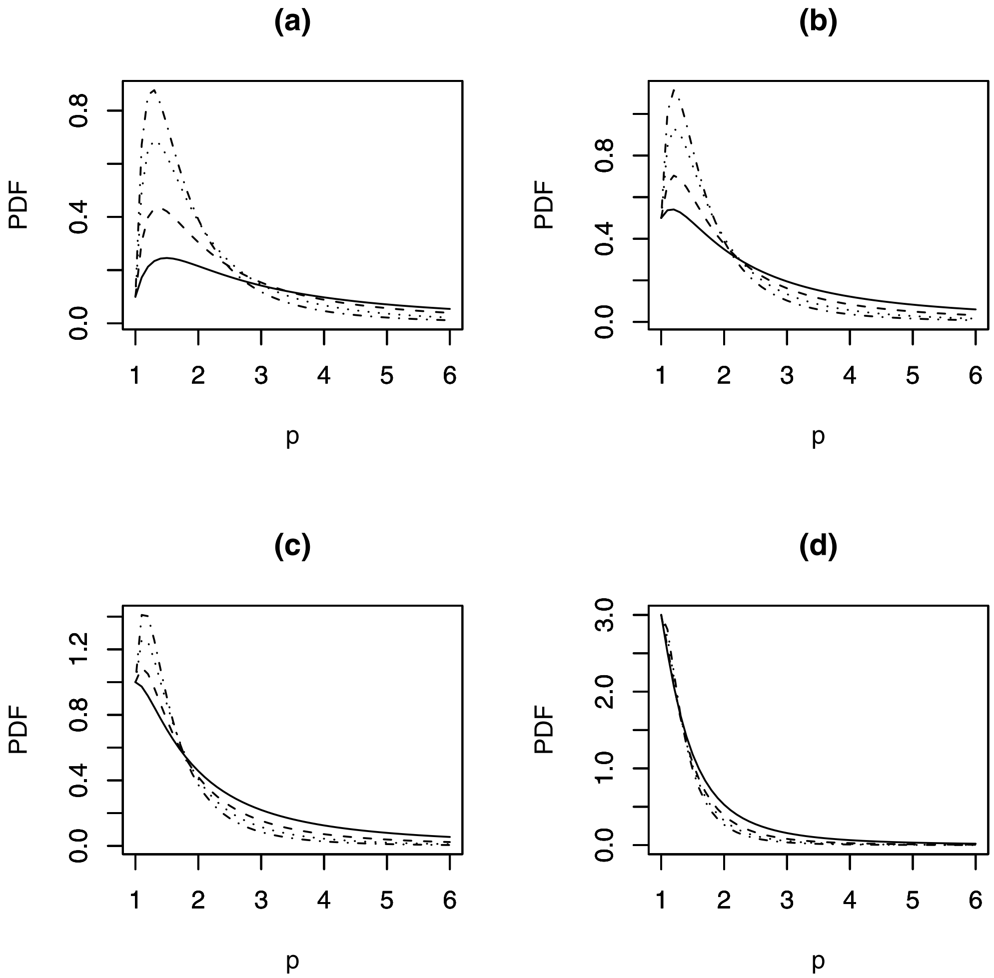

Figure 2: *Plots of the pdf of (11) for* $\beta = 1$ *and (a):* $\lambda_0 = 0.1$*; (b):* $\lambda_0 = 0.5$*; (c):* $\lambda_0 = 1$*; and, (d):* $\lambda_0 = 3$. *The four curves in each plot are: the solid curve (*$\lambda_1 = 1$, $\lambda_2 = 1$*), the curve of lines (*$\lambda_1 = 3$, $\lambda_2 = 1$*), the curve of dots (*$\lambda_1 = 3$, $\lambda_2 = 2$*), and the curve of lines and dots (*$\lambda_1 = 3$, $\lambda_2 = 3$*).*

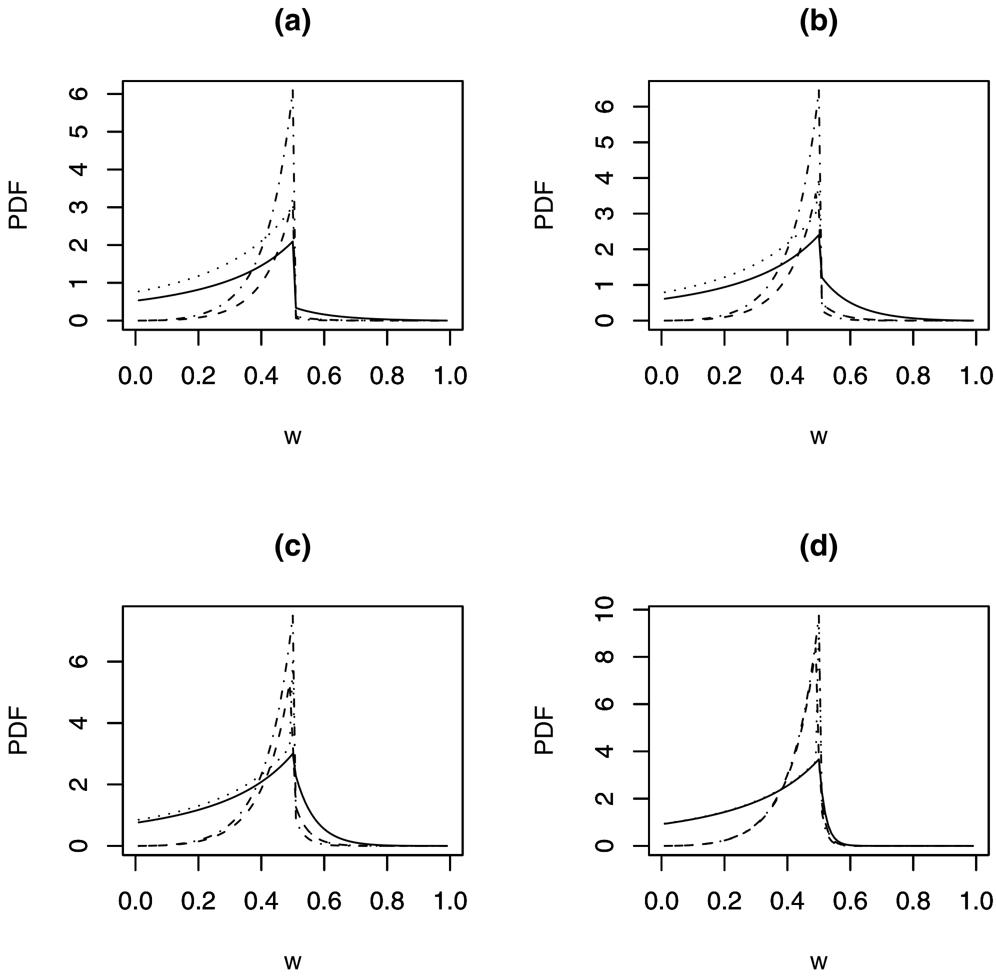

Figure 3: *Plots of the pdf of (12) for* $\beta = 1$ *and (a):* $\lambda_0 = 0.1$*; (b):* $\lambda_0 = 0.5$*; (c):* $\lambda_0 = 2$*; and, (d):* $\lambda_0 = 10$. *The four curves in each plot are: the solid curve* ($\lambda_1 = 1$, $\lambda_2 = 1$)*, the curve of lines* ($\lambda_1 = 1$, $\lambda_2 = 3$)*, the curve of dots* ($\lambda_1 = 3$, $\lambda_2 = 1$)*, and the curve of lines and dots* ($\lambda_1 = 3$, $\lambda_2 = 3$)*.*

3 Moments

Here, we derive the moments of $R = X + Y$, $P = XY$ and $W = X/(X + Y)$ when X and Y are distributed according to (1)–(3). We need the following lemma.

Lemma 2 *If X and Y are jointly distributed according to (1)–(3) then*

$$\begin{aligned} E\left(X^m Y^n\right) &= \frac{\beta^{m+n}\lambda_2\left(\lambda_0+\lambda_1\right)\left(\lambda_1+\lambda_2\right)}{\left(m-\lambda_0-\lambda_1\right)\left(m+n-\lambda_0-\lambda_1-\lambda_2\right)\left(\lambda_0+\lambda_1+\lambda_2\right)} \\ &\quad + \frac{\beta^{m+n}\lambda_1\left(\lambda_0+\lambda_2\right)\left(\lambda_1+\lambda_2\right)}{\left(n-\lambda_0-\lambda_2\right)\left(m+n-\lambda_0-\lambda_1-\lambda_2\right)\left(\lambda_0+\lambda_1+\lambda_2\right)} \\ &\quad + \frac{\beta^{m+n}\lambda_0}{m+n+\lambda_0+\lambda_1+\lambda_2} \end{aligned}$$

for $m \geq 1$ *and* $n \geq 1$.

Proof. One can write

$$\begin{aligned} E\left(X^m Y^n\right) &= \frac{\lambda_2\left(\lambda_0+\lambda_1\right)\left(\lambda_1+\lambda_2\right)}{\lambda_0+\lambda_1+\lambda_2}\beta^{\lambda_0+\lambda_1+\lambda_2}\int_\beta^\infty\int_y^\infty x^{m-1-\lambda_0-\lambda_1}y^{n-1-\lambda_2}dxdy \\ &\quad +\lambda_0\beta^{\lambda_0+\lambda_1+\lambda_2}\int_\beta^\infty x^{m+n-1-\lambda_0-\lambda_1-\lambda_2}dx \\ &\quad + \frac{\lambda_1\left(\lambda_0+\lambda_2\right)\left(\lambda_1+\lambda_2\right)}{\lambda_0+\lambda_1+\lambda_2}\beta^{\lambda_0+\lambda_1+\lambda_2}\int_\beta^\infty\int_x^\infty y^{-(1+\lambda_0+\lambda_2)}x^{-(1+\lambda_1)}dydx. \end{aligned}$$

The result of the theorem follows by elementary integration of the above integrals. ■

The moments of $R = X + Y$ and $P = XY$ are now simple consequences of this lemma as illustrated in Theorems 4 and 5. The moments of $W = X/(X + Y)$ require a separate treatment as shown by Theorem 6.

Theorem 4 *If X and Y are jointly distributed according to (1)–(3) then*

$$\begin{aligned} E\left(R^n\right) &= \frac{2^n\beta^n\lambda_0}{n+\lambda_0+\lambda_1+\lambda_2} + \sum_{k=0}^{n}\binom{n}{k}\left[\frac{\beta^n\lambda_2\left(\lambda_0+\lambda_1\right)\left(\lambda_1+\lambda_2\right)}{\left(n-k-\lambda_0-\lambda_1\right)\left(n-\lambda_0-\lambda_1-\lambda_2\right)\left(\lambda_0+\lambda_1+\lambda_2\right)}\right. \\ &\quad \left. + \frac{\beta^n\lambda_1\left(\lambda_0+\lambda_2\right)\left(\lambda_1+\lambda_2\right)}{\left(k-\lambda_0-\lambda_2\right)\left(n-\lambda_0-\lambda_1-\lambda_2\right)\left(\lambda_0+\lambda_1+\lambda_2\right)}\right] \end{aligned}$$

for $n \geq 1$.

Proof. the result follows by writing

$$E\left((X+Y)^n\right) = \sum_{k=0}^{n} \binom{n}{k} E\left(X^{n-k} Y^k\right)$$

and applying Lemma 2 to each expectation in the sum. ■

Theorem 5 *If X and Y are jointly distributed according to (1)–(3) then*

$$\begin{aligned} E\left(P^n\right) &= \frac{\beta^{2n}\lambda_2\left(\lambda_0+\lambda_1\right)\left(\lambda_1+\lambda_2\right)}{\left(n-\lambda_0-\lambda_1\right)\left(2n-\lambda_0-\lambda_1-\lambda_2\right)\left(\lambda_0+\lambda_1+\lambda_2\right)} \\ &\quad + \frac{\beta^{2n}\lambda_1\left(\lambda_0+\lambda_2\right)\left(\lambda_1+\lambda_2\right)}{\left(n-\lambda_0-\lambda_2\right)\left(2n-\lambda_0-\lambda_1-\lambda_2\right)\left(\lambda_0+\lambda_1+\lambda_2\right)} \\ &\quad + \frac{\beta^{2n}\lambda_0}{2n+\lambda_0+\lambda_1+\lambda_2} \end{aligned}$$

for $n \geq 1$.

Proof. follows by writing $E(P^n) = E(X^n Y^n)$ and applying Lemma 2 with $m = n$. ■

Theorem 6 *If X and Y are jointly distributed according to (1)–(3) then*

$$\begin{aligned} E\left(W^n\right) &= \frac{2^{1-n}\lambda_0}{\lambda_0+\lambda_1+\lambda_2} + \frac{\lambda_2\left(\lambda_0+\lambda_1\right)\left(\lambda_1+\lambda_2\right)}{\left(\lambda_0+\lambda_1+\lambda_2\right)^2} B_{1/2}\left(\lambda_0+\lambda_1, n-\lambda_0-\lambda_1\right) \\ &\quad + \frac{\lambda_1\left(\lambda_0+\lambda_2\right)\left(\lambda_1+\lambda_2\right)}{\left(\lambda_0+\lambda_1+\lambda_2\right)^2} B_{1/2}\left(n+\lambda_0+\lambda_2, -\lambda_0-\lambda_2\right) \end{aligned} \tag{14}$$

for $n \geq 1$.

Proof. Using (12), one can write

$$\begin{aligned} E\left(W^n\right) &= \frac{\lambda_2\left(\lambda_0+\lambda_1\right)\left(\lambda_1+\lambda_2\right)}{\left(\lambda_0+\lambda_1+\lambda_2\right)^2} \int_{1/2}^{1} \frac{w^n(1-w)^{\lambda_0+\lambda_1-1}}{w^{\lambda_0+\lambda_1+1}} dw + \frac{2^{1-n}\lambda_0}{\lambda_0+\lambda_1+\lambda_2} \\ &\quad + \frac{\lambda_1\left(\lambda_0+\lambda_2\right)\left(\lambda_1+\lambda_2\right)}{\left(\lambda_0+\lambda_1+\lambda_2\right)^2} \int_{0}^{1/2} \frac{w^{n+\lambda_0+\lambda_2-1}}{(1-w)^{\lambda_0+\lambda_2+1}} dw. \end{aligned}$$

The result of the theorem follows by the definition of the incomplete beta function. ■

Using special properties of the incomplete beta function, one can derive elementary forms of (14). This is shown in the corollaries below.

Corollary 4 *If X and Y are jointly distributed according to (1)–(3) and if $\lambda_0 + \lambda_1 \geq 1$ is an integer then $E(W^n)$ is given by (14) with*

$$B_{1/2}(\lambda_0 + \lambda_1, n - \lambda_0 - \lambda_1) = 1 - 2^{\lambda_0+\lambda_1-n} \sum_{k=1}^{\lambda_0+\lambda_1} \frac{\Gamma(n - \lambda_0 - \lambda_1 + k - 1)}{\Gamma(n - \lambda_0 - \lambda_1)\Gamma(k)} 2^{1-k}.$$

Corollary 5 *If X and Y are jointly distributed according to (1)–(3) and if $\lambda_0 + \lambda_2 \geq 1$ is an integer then $E(W^n)$ is given by (14) with*

$$B_{1/2}(n + \lambda_0 + \lambda_2, -\lambda_0 - \lambda_2) = 1 - 2^{\lambda_0+\lambda_2} \sum_{k=1}^{n+\lambda_0+\lambda_2} \frac{\Gamma(n - \lambda_0 - \lambda_2 - 1)}{\Gamma(-\lambda_0 - \lambda_2)\Gamma(k)} 2^{1-k}.$$

Percentage points for $R = X + Y$

λ_0	λ_1	λ_2	α						
			0.9	0.95	0.975	0.99	0.995	0.999	0.9995
1	1	1	9.62379	16.49151	29.78508	70.40482	137.2642	662.6351	1259.997
1	1	2	6.38258	9.402676	14.83461	30.03895	55.16546	260.3903	511.3611
1	1	3	5.419237	7.756518	12.03189	24.47247	45.52252	212.9589	429.7302
1	1	4	5.028103	7.113937	10.81814	21.20653	38.43352	183.8694	342.4035
1	2	1	6.390126	9.439726	14.88197	30.05685	54.93929	265.512	551.438
1	2	2	4.709026	5.991161	7.717563	11.07508	14.77118	30.48835	43.03218
1	2	3	4.108536	5.012974	6.194804	8.438637	10.91330	21.23383	29.51412
1	2	4	3.806872	4.581755	5.616983	7.630785	9.834232	19.15476	25.87484
1	3	1	5.435317	7.754701	11.96454	23.94088	44.06230	218.9689	440.4970
1	3	2	4.104989	5.016544	6.21157	8.413445	10.88446	21.34119	29.27734
1	3	3	3.617398	4.213916	4.936576	6.121482	7.261394	11.20831	13.68865
1	3	4	3.369725	3.85984	4.433214	5.385524	6.294604	9.31356	11.19996
1	4	1	5.011635	7.096623	10.78617	21.23060	38.18032	175.2684	338.8366
1	4	2	3.80762	4.58071	5.625067	7.593149	9.829113	18.83798	25.31333
1	4	3	3.372456	3.862036	4.436062	5.373236	6.295974	9.400779	11.25767
1	4	4	3.149176	3.525237	3.951599	4.617037	5.197676	7.02943	7.937064
2	1	1	6.908587	11.80303	21.87108	52.17421	103.0996	499.1869	997.795
2	1	2	5.026371	7.20023	11.28552	23.11933	42.80327	195.6307	377.0428
2	1	3	4.290314	5.792941	8.742232	18.13929	34.57562	164.6154	310.1243
2	1	4	3.932082	5.174204	7.661537	16.03917	29.84287	150.3417	292.0324
2	2	1	5.022213	7.195747	11.29309	23.38030	43.37083	202.7843	408.1873
2	2	2	4.13171	5.197745	6.670142	9.64867	12.99950	27.08907	37.89176
2	2	3	3.688318	4.443407	5.44835	7.400021	9.607725	18.71532	25.11885
2	2	4	3.446081	4.072247	4.913453	6.573777	8.551287	17.07842	23.09799
2	3	1	4.29382	5.793729	8.750932	18.24647	35.13249	172.9237	343.8436
2	3	2	3.695045	4.449435	5.443715	7.428022	9.612759	19.03360	25.71027
2	3	3	3.365524	3.891746	4.532711	5.631077	6.69489	10.43221	12.59107
2	3	4	3.170085	3.599843	4.105821	4.970653	5.800506	8.645942	10.33633
2	4	1	3.938950	5.184407	7.665587	15.86660	30.17476	143.1466	284.7486

2	4	2	3.444092	4.074746	4.925434	6.631013	8.602288	17.32115	23.43268
2	4	3	3.174564	3.601501	4.110391	4.961649	5.79828	8.581511	10.37010
2	4	4	3.003146	3.33961	3.724340	4.33036	4.89138	6.589682	7.558714
3	1	1	5.556553	9.369284	17.22973	41.75966	82.96045	402.7553	777.0871
3	1	2	4.356332	6.135416	9.57303	19.70144	36.90099	170.3009	324.3305
3	1	3	3.798096	5.011188	7.527998	15.77983	29.90835	145.9120	283.4977
3	1	4	3.499727	4.460710	6.534267	13.78972	26.22162	132.2372	284.8490
3	2	1	4.346251	6.141804	9.52928	19.51615	35.56617	174.0592	359.7555
3	2	2	3.795461	4.756487	6.126913	8.92387	12.12327	25.40350	35.55154
3	2	3	3.446532	4.123285	5.055806	6.904248	9.032536	17.89067	24.55036
3	2	4	3.234209	3.78897	4.564601	6.174418	8.080741	16.08926	22.26428
3	3	1	3.797431	5.014095	7.522061	15.69015	29.62894	147.5539	316.1749
3	3	2	3.445716	4.130013	5.054499	6.874383	9.011033	17.91510	24.94480
3	3	3	3.199305	3.686841	4.28561	5.317371	6.38796	10.10062	12.19057
3	3	4	3.035728	3.430755	3.906324	4.714228	5.508655	8.228773	9.973009
3	4	1	3.491053	4.432245	6.497168	13.59319	25.43914	123.7052	239.6344
3	4	2	3.234879	3.797424	4.571231	6.147995	8.042847	16.38848	22.63103
3	4	3	3.036627	3.430917	3.905991	4.727645	5.506118	8.309395	9.892161
3	4	4	2.905312	3.219497	3.58608	4.175939	4.700929	6.402495	7.481372
4	1	1	4.751449	8.002954	14.64670	34.75727	67.59802	329.042	666.0516
4	1	2	3.928277	5.493376	8.457637	17.04472	31.56688	148.9422	289.8442
4	1	3	3.496069	4.542001	6.688109	13.90893	26.0087	119.8621	241.3712
4	1	4	3.244412	4.050198	5.87665	12.53929	24.00743	113.0687	238.5718
4	2	1	3.941387	5.541231	8.624192	17.4132	32.66607	151.8941	300.2353
4	2	2	3.552877	4.440468	5.732454	8.303123	11.28017	24.11745	34.02633
4	2	3	3.273833	3.909206	4.794988	6.536048	8.465026	16.80705	23.01364
4	2	4	3.089486	3.605092	4.332355	5.847722	7.649926	15.64277	21.51948
4	3	1	3.490893	4.541283	6.738373	14.01726	26.40212	125.2505	243.3163
4	3	2	3.269694	3.89497	4.77076	6.543371	8.531305	17.14365	23.55384
4	3	3	3.077487	3.536144	4.113783	5.12258	6.14586	9.683466	11.84106
4	3	4	2.937028	3.308072	3.77329	4.567861	5.368209	8.138209	9.9069
4	4	1	3.246128	4.043495	5.834828	12.46214	23.63761	113.7882	219.6971
4	4	2	3.08685	3.599695	4.323378	5.837564	7.619	15.45620	21.34394
4	4	3	2.935896	3.307082	3.767617	4.569923	5.372604	8.09562	9.654783
4	4	4	2.826395	3.128217	3.478607	4.048215	4.57136	6.206746	7.152812

4 Percentiles

In this section, we provide extensive tabulations of the percentiles of the distribution of R (percentiles for P and W are not given since their pdfs are elementary). These percentiles are computed numerically by solving the equation

$$\int_0^{r_\alpha} f_R(r) dr \quad = \quad \alpha,$$

where $f_R(r)$ is given by (4). Evidently, this involves computation of the hypergeometric functions and routines for this are widely available. We used the function hypergeom $(\cdot)$ in the algebraic manipulation package, MAPLE. The percentiles are given for $\alpha = 0.90$, 0.95, 0.975, 0.99, 0.995, 0.999, 0.9995, $\beta = 1$, $\lambda_0 = 1, 2, 3, 4$, $\lambda_1 = 1, 2, 3, 4$ and $\lambda_2 = 1, 2, 3, 4$.

Similar tabulations could be easily derived for other values of λ_0, λ_1 and λ_2. We hope these numbers will be of use to the practitioners of the bivariate Pareto distribution (see Section 1).

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