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A note on interval estimation for the mean of inverse Gaussian distribution

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Abstract

In this paper, we study the interval estimation for the mean from inverse Gaussian distribution. This distribution is a member of the natural exponential families with cubic variance function. Also, we simulate the coverage probabilities for the confidence intervals considered. The results show that the likelihood ratio interval is the best interval and Wald interval has the poorest performance.

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1 Introduction

Interval estimation for natural exponential family (NEF) is discussed in many references for more than sixty years. Santner (1998), Agresti and Coull (1998), Brown, Cai and DasGupta (2001, 2002) considered Wald score, Agresti-Coull, likelihood ratio and Jeffreys intervals. Also, a good contribution for comparison of these intervals for NEF with quadratic variance function is given by Brown, Cai and DasGupta (2003). Several alternative intervals, coverage probability and length expansion of the Wald, score,

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likelihood ratio, Agresti-Coull and Jeffreys intervals are studied, simulated and approximated in Brown, Cai and DasGupta (2003) for normal, gamma, generalized hyperbolic secant (GHS), binomial, negative binomial and Poisson as the member of NEF with quadratic variance function. In this paper, we extend several intervals for inverse Gaussian distribution as a member of NEF with cubic variance function (CVF). Also, we show the coverage probabilities of these intervals via a simulation study, while noting that the theoretical approach for it is difficult, because finding the roots of a polynomial of degree 3 equation is not always easy. Applying the results for other NEF-CVF, especially for Abel and Takacs distributions, is a topic for future study.

This paper is organized as follows: In Section 2, we introduce the natural exponential family with quadratic and cubic variance functions. In Section 3, we study the confidence intervals for the mean of the inverse Gaussian as the most famous member of the natural exponential family with cubic variance function. In Section 4, the coverage probabilities for the confidence intervals are simulated in two case: with the fixed sample size (n), fixed λ and the variable parameter μ, and with the fixed parameters and the variable sample size. In Section 5, the proposed confidence intervals have been applied for the mean of ball bearings. A brief conclusion is provided in Section 6.

2 Natural exponential family

The distributions in a natural exponential family (NEF) have the following form

$$f(x;\xi) = \exp(\xi x - \psi(\xi))h(x),$$

where ξ is called the natural parameter. The mean, variance and cumulant generating function for this family are

$$\begin{cases} \mu = E(X) = \psi'(\xi), \\ \sigma^2 = Var(X) = \psi''(\xi), \\ \Phi(t) = \psi(t+\xi) - \psi(\xi). \end{cases}$$

In a subclass, on the real line, the variance function is a polynomial function with degree lower than or equal to 3. We get twelve different types, the first six appear in Morris (1982,1983), Shanbhag (1972, 1979) and Brown (1986). The others introduced in Letac and Mora (1990), where the inverse Gaussian distribution is one of the most well-known members of NEF-CVF. The variance $\psi''(\xi)$ of NEF-CVF depends on ξ only through the mean μ, as

$$\sigma^2 = Var(X) = a_0 + a_1\mu + a_2\mu^2 + a_3\mu^3,$$

where a_0, a_1, a_2, and a_3 are suitable constants. The NEF-QVF is the special case with $a_3 = 0$. NEF-QVF consists of six important distributions: normal, gamma, generalized hyperbolic secant (GHS), binomial, negative binomial, and Poisson (see Morris (1982,1983) and Brown (1986)). Also, NEF-CVF consists of these six distributions: Ressel, inverse Gaussian, Abel, Takacs, strict arcsine, and large arcsine. We focus our attentions on the inverse Gaussian distribution, $IG(\mu, \lambda)$, with density

$$f(x) = \sqrt{\frac{\lambda}{2\pi x^3}} \exp\left(-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right), \qquad x > 0, \qquad \mu > 0, \qquad \lambda > 0.$$

For this distribution, we have:

$$\xi = \frac{-\lambda}{2\mu^2}, \qquad \psi(\xi) = -\sqrt{(-2\lambda\xi)}, \qquad h(x) = \sqrt{\frac{\lambda}{2\pi x^3}} \exp(\frac{-\lambda}{2x}).$$

Hence, $Var(X) = \frac{\mu^3}{\lambda}$ with $a_0 = a_1 = a_2 = 0$, and $a_3 = \frac{1}{\lambda}$.

3 Confidence intervals for the mean of inverse Gaussian distribution

Based on a random sample of size n from $IG(\mu, \lambda)$ (λ known), we want to calculate some of the confidence intervals, at the confidence level $1-\alpha$, for mean μ. We also know that for this distribution the MLE of μ is the sample mean, that is $\hat{\mu} = \bar{X}$. We can construct the following confidence intervals for μ.

Definition 1 *The* ***Wald interval*** *(CI_W) is based on Slutsky's statistic* $W_n = \frac{\sqrt{n}(\hat{\mu}-\mu)}{\hat{\sigma}}$ *as*

$$CI_W : \hat{\mu} \pm k(n\lambda)^{-1/2}\hat{\mu}^{3/2},$$

where $k = z_{1-\alpha/2}$.

Definition 2 *The* ***score interval*** *(CI_S) is based on the Central Limit theorem with* $Z = \frac{\sqrt{n}(\hat{\mu}-\mu)}{\sigma}$. *We construct this interval by inverting Rao's equal-tailed score test of* $H_0 : \mu = \mu_0$. *Hence, we accept* H_0 *based on Rao's score test if and only if* μ_0 *is in the interval. We solve a cubic equation in terms of* μ, *under the following relation:*

$$-k \leq \frac{\sqrt{n\lambda}(\hat{\mu}-\mu)}{\sqrt{\mu^3}} \leq k.$$

By substituting $\mu = 1/x^2$ *in the above equations, we have*

$$\begin{cases} x^3 - \dfrac{x}{\hat{\mu}} - \dfrac{k}{\sqrt{n\lambda}\hat{\mu}} \leq 0, & (1) \\ x^3 - \dfrac{x}{\hat{\mu}} + \dfrac{k}{\sqrt{n\lambda}\hat{\mu}} \geq 0. & (2) \end{cases}$$

Suppose that, $Q = \dfrac{1}{3\hat{\mu}}$, $R_1 = \dfrac{-k}{2\sqrt{n\lambda}\hat{\mu}}$, *and* $R_2 = \dfrac{k}{2\sqrt{n\lambda}\hat{\mu}}$, *then we have the following cases:*

• ***Case 1:*** *If* $4n\lambda - 27\hat{\mu}k^2 < 0$, *then* $Q^3 - R_1^2 < 0$. *Hence, the relations* (1) *and* (2) *imply*

$$\begin{cases} x \leq [(R_2 + \sqrt{R_2^2 - Q^3})^{1/3} + (R_2 - \sqrt{R_2^2 - Q^3})^{1/3}] = M, \\ x \geq -[(R_2 + \sqrt{R_2^2 - Q^3})^{1/3} + (R_2 - \sqrt{R_2^2 - Q^3})^{1/3}] = -M. \end{cases}$$

By substituting $\mu = 1/x^2$ *in the above equations, we obtain a one-sided interval for* μ *as* $\dfrac{1}{M^2} \leq \mu$.

• ***Case 2:*** *If* $4n\lambda - 27\hat{\mu}k^2 \geq 0$, *then* $Q^3 - R_1^2 \geq 0$. *The roots of the equations* (1) *and* (2) *are as*

$$\begin{cases} x_1 = -2\sqrt{Q}\cos\left(\dfrac{\theta_1}{3}\right), \\ x_2 = -2\sqrt{Q}\cos\left(\dfrac{\theta_1 + 2\pi}{3}\right), \\ x_3 = -2\sqrt{Q}\cos\left(\dfrac{\theta_1 + 4\pi}{3}\right), \end{cases} \qquad \begin{cases} x'_1 = -2\sqrt{Q}\cos\left(\dfrac{\theta_2}{3}\right), \\ x'_2 = -2\sqrt{Q}\cos\left(\dfrac{\theta_2 + 2\pi}{3}\right), \\ x'_3 = -2\sqrt{Q}\cos\left(\dfrac{\theta_2 + 4\pi}{3}\right), \end{cases}$$

where $\theta_1 = \arccos\left(\dfrac{R_1}{\sqrt{Q^3}}\right)$ *and* $\theta_2 = \arccos\left(\dfrac{R_2}{\sqrt{Q^3}}\right)$. *Based on the above roots and by substituting* $\mu = \dfrac{1}{x^2}$, *the Score interval is calculated as*

$$\begin{cases} \frac{1}{\sqrt{x'_3}} \leq \mu, \\ \dfrac{1}{\sqrt{x_2}} \leq \mu \leq \dfrac{1}{\sqrt{x'_2}}. \end{cases}$$

Definition 3 *The* ***likelihood ratio interval*** (CI_{LR}) *is constructed by inverting the likelihood ratio test under* $H_0 : \mu = \mu_0$ *(see Rao (1973) and Serfling (1980)). The likelihood ratio* $\Lambda_n = \frac{L(\mu_0)}{\sup_\mu L(\mu)}$ *for inverse Gaussian distribution is given by*

$$\Lambda_n = \exp(\frac{-n\lambda\hat{\mu}}{2\mu^2} + \frac{n\lambda}{\mu} - \frac{n\lambda}{2\hat{\mu}}).$$

By solving the equation $-2\log\Lambda_n \leq \chi^2_{\alpha,1} = k^2$ *in terms of* μ*, the likelihood ratio interval for* μ *is calculated as*

$$\frac{n\lambda\hat{\mu}}{n\lambda + k\sqrt{n\lambda\hat{\mu}}} \leq \mu \leq \frac{n\lambda\hat{\mu}}{n\lambda - k\sqrt{n\lambda\hat{\mu}}},$$

where $0 < k < \sqrt{\frac{n\lambda}{\hat{\mu}}}$.

Note: If λ in inverse Gaussian distribution is unknown, then we substitute a point estimation instead of λ in the above equations (for example MLE $\hat{\lambda} = n\left[\sum_{i=1}^{n}\frac{(x_i - \overline{x})^2}{x_i\overline{x}^2}\right]^{-1}$).

4 The coverage probability via a simulation study

In this section, we estimate by simulation the coverage probabilities of the considered confidence intervals for the mean. Suppose that, the parameters μ and λ of inverse Gaussian distribution are fixed values. Based on a random sample $X_1, X_2, \ldots, X_n$ of $IG(\mu, \lambda)$, we calculate the confidence intervals with $\hat{\lambda} = n\left[\sum_{i=1}^{n}\frac{(x_i - \overline{x})^2}{x_i\overline{x}^2}\right]^{-1}$, at the confidence level $1 - \alpha = 0.95$, using 1000 repetitions for each simulation set. The estimated coverage probability (ECP) is the proportion of confidence intervals containing the fixed value of the mean.

The estimation of the coverage probability is calculated by simulation in two cases: with fixed parameters and variable sample size ($\lambda = 231.711$, $\mu = 72.219$ and $5 \leq n \leq 100$), and with fixed sample size ($n = 23$), fixed $\lambda = 10$ and variable $0 < \mu \leq 6$ (see Fig. 1 and Fig. 2).

From Fig. 1 and Fig. 2, we can refer the following results for the confidence intervals in inverse Gaussian distribution:

- The averages of the estimated coverage probabilities (ECP) for the considered confidence intervals, are organized as $ECP_W \leq ECP_S \leq ECP_{LR}$.

- The averages of the estimated coverage probabilities for Wald, score, and likelihood ratio intervals are lower than the confidence level.

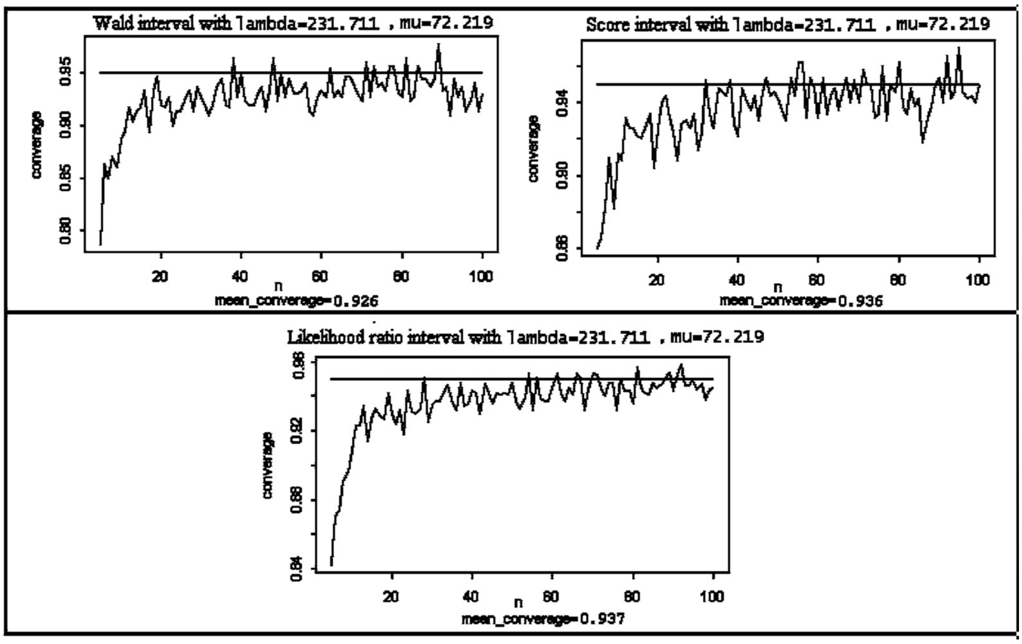

Figure 1: *Coverage probability with fixed parameters* (λ, μ) *and variable n from 5 to 100.*

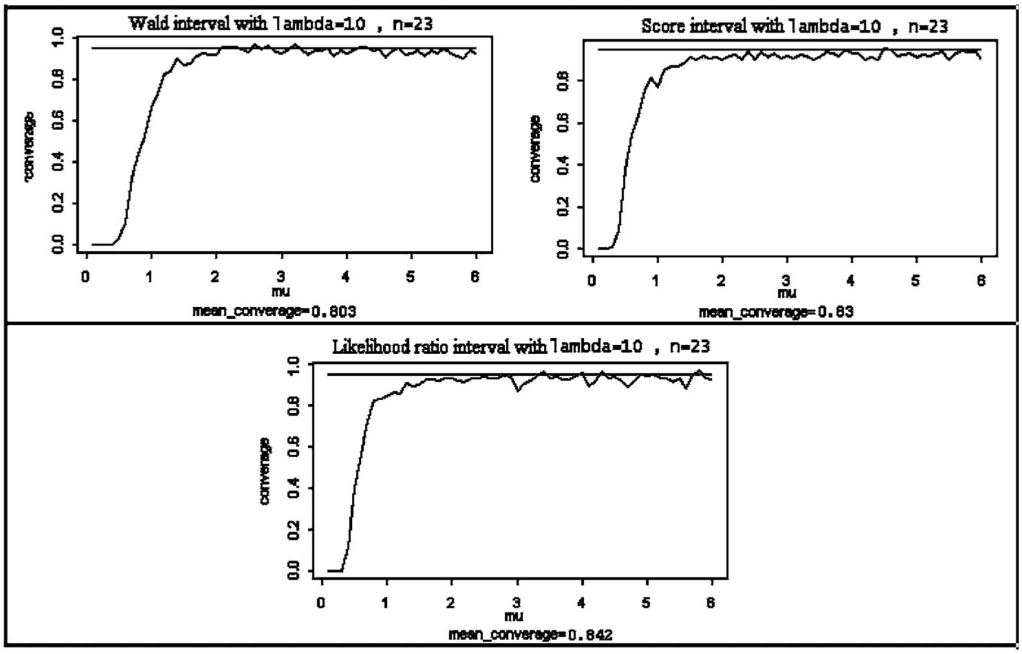

Figure 2: *Coverage probability with fixed n, fixed* λ *and variable parameter* μ*.*

- When parameter μ goes to zero, the estimated coverage probability decreases.
- When the sample size increases, the average of coverage probabilities tends to the confidence level.

5 Application of the confidence intervals from an inverse Gaussian distribution

Twenty three ball bearings were used in the life test study and yielded the following in millions of revolutions to failure, (see Pavur *et al.*, 1992).

Data	Data	Data	Data	Data	Data
17.88	51.96	93.12	28.92	54.12	98.64
33.00	55.56	105.12	41.52	67.80	105.84
42.12	68.64	127.92	45.60	68.64	127.92
48.48	68.88	173.40	51.84	84.12	

They showed that the data was from an inverse Gaussian population. We use the data as well to demonstrate our confidence intervals. We estimate $\hat{\mu} = 72.219$, and $\hat{\lambda} = 231.711$. The confidence intervals, at the confidence level $1 - \alpha = 0.95$, for the mean are calculated as:

$$\begin{cases} CI_W = [55.74, 88.70], \\ CI_S = [59.80, 98.44], \\ CI_{LR} = [58.80, 93.57]. \end{cases}$$

The lengths of the three considered confidence intervals are:

$$L_W = 32.95, \qquad L_S = 38.64, \qquad L_{LR} = 34.77.$$

Hence, $L_W < L_{LR} < L_S$. Based on a random sample of size $n = 23$ with $\lambda = 231.711$ and $\mu = 72.219$, the estimated the coverage probabilities (ECP) for Wald, score, and likelihood ratio intervals are calculated by simulation as (see Fig. 1):

$$ECP_W = 0.922, \qquad ECP_S = 0.933, \qquad ECP_{LR} = 0.935.$$

Because of Wald interval is uniformly poor, we prefer to use of the likelihood ratio interval. The likelihood ratio interval is better than the score interval of the viewpoint its coverage probability and length ($ECP_S < ECP_{LR}$ and $L_{LR} < L_S$). Also, we can use of the above confidence intervals for testing hypothesis for the mean of ball bearings.

6 Conclusion

In this paper, we calculate Wald score, and likelihood ratio intervals for inverse Gaussian distribution. Also, we estimate the coverage probabilities by simulation. The simulation study shows that the Wald confidence interval has the lowest estimated coverage probability. The likelihood ratio interval is better, having an estimated coverage probability very close to the nominal value. Finally we apply the results for an example (ball bearings).

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