SORT 35 (2) July-December 2011, 145-164

Iterative beam search for simple assembly line balancing with a fixed number of work stations

Christian Blum*

Abstract

The simple assembly line balancing problem (SALBP) concerns the assignment of tasks with pre-defined processing times to work stations that are arranged in a line. Hereby, precedence constraints between the tasks must be respected. The optimization goal of the SALBP-2 variant of the problem concerns the minimization of the so-called cycle time, that is, the time in which the tasks of each work station must be completed. In this work we propose to tackle this problem with an iterative search method based on beam search. The proposed algorithm is able to generate optimal solutions, respectively the best upper bounds, for 283 out of 302 test cases. Moreover, for 9 further test cases the algorithm is able to improve the currently best upper bounds. These numbers indicate that the proposed iterative beam search algorithm is currently a state-of-the-art method for the SALBP-2.

MSC2000 Classification: 90C27 (Combinatorial Optimization)

Keywords: Assembly line balancing, fixed number of work stations, beam search.

1. Introduction

The class of problems known as assembly line balancing problems (ALBPs) concerns the optimization of processes related to the manufacturing of products via assembly lines. Their importance in the industrial world is shown by the fact that much research efforts have been dedicated to many different types of ALBPs during the past 50-60 years Gosh and Gagnon (1989), Salveson (1955). The specific problem considered in this paper is the so-called simple assembly line balancing problem (SALBP) Scholl and Becker (2006), a well-studied scientific test case. An assembly line is composed of a set of work stations arranged in a line, and by a transport system which moves the product to be manufactured along the line. The product is manufactured by executing a given

**Address for correspondence:* ALBCOM Research Group. Universitat Politècnica de Catalunya. C/ Jordi Girona 1-3, Campus Nord, Omega 112. 08034 Barcelona (Spain). cblum@lsi.upc.edu.
Received: December 2010
Accepted: October 2011

set of tasks. Each of these tasks has a pre-defined processing time. In order to obtain a solution to a given SALBP instance, all tasks must be assigned to work stations subject to precedence constraints between the tasks. In the context of the SALBP, all work stations are considered to be of equal size. Moreover, the assembly line is assumed to move at a constant speed. This implies a maximum of C time units – the so-called *cycle time* – for processing the tasks assigned to each work station. The SALBP has been tackled with several objective functions among which the following ones are the most studied ones in the literature:

- Given a fixed cycle time C, the optimization goal consists in minimizing the number of necessary work stations. This variant of the problem is refered to as SALBP-1.
- Given a fixed number m of work stations, the goal is to minimize the cycle time C. The literature knows this second problem variant as SALBP-2.

The feasibility problem SALBP-F arises when both a cycle time C and a number of work stations m is given and the goal is to find a feasible solution respecting C and m. In this work we will deal with the SALBP-2 variant of the problem.

For what concerns the comparison between SALBP-1 and SALBP-2, much of the scientific work has been dedicated to the SALBP-1. However, also for the SALBP-2 exists a considerable body of research papers. An excellent survey was provided by Scholl and Becker (2006). Approaches for the SALBP-2 can basically be classified as either *iterative solution approaches* or *direct solution approaches*. Iterative approaches tackle the problem by iteratively solving a series of SALBP-F problems that are obtained by fixing the cycle time. This process is started with a cycle time that is set to some calculated upper bound. This cycle time is then decremented during the iterative process, which stops as soon as no solution for the corresponding SALBP-F problem can be found. In contrast to these indirect approaches, direct approaches intend to solve a given SALBP-2 instance directly.

Heuristic as well as exact approaches have been devised for the SALBP-2. Among the existing exact methods we find iterative approaches such as the ones proposed in Hackman et al. (1989), Scholl (1999) but also direct approaches such as the ones described in Klein and Scholl (1996), Scholl (1994). Moreover, the performance of different integer programming formulations of the SALBP-2 have been evaluated in Pastor and Ferrer (2009), Pastor et al. (2007). The currently best-performing exact method is SALOME-2 Klein and Scholl (1996), which is a branch & bound procedure making use of a so-called local lower bound method (LLBM), a bidirectional branching strategy, and several dominance and reduction rules. Surprisingly, exact methods outperform the existing heuristic and metaheuristic approaches for the SALBP-2. While 287 (out of 302) existing problem instances were solved to optimality by exact approaches, the most successful metaheuristic approach to date—a tabu search method proposed in Scholl and Voss (1996) —was only able to find the best upper bounds with respect to 168 prob-

lem instances. Note that SALOME-2 alone was able to solve 217 problem instances to optimality. Apart from the above-mentioned tabu search, another tabu search proposal can be found in Chiang (1998). Other metaheuristic approaches include evolutionary algorithms Anderson and Ferris (1994), Nearchou (2007), Watanabe et al. (1995) and simulated annealing Henrici (1994). Moreover, a two-phase heuristic based on linear programming can be found in Ugurdag et al. (1997), whereas a heuristic based on Petri nets was proposed in Kilincci (2010). Finally, an approach similar to the one proposed in this paper has been presented in Blum and Miralles (2011) for a more general problem, the assembly line worker assignment and balancing problem (ALWABP).

Contribution of this work. Subsequently we propose to tackle the SALBP-2 by means of an iterative approach based on beam search, which is an incomplete variant of branch & bound. The resulting iterative beam search algorithm is inspired by one of the current state-of-the-art methods for the SALBP-1, namely Beam-ACO Blum (2008). Beam-ACO is a hybrid approach that is obtained by combining the metaheuristic ant colony optimization with beam search. In this work we propose to use the beam search component of Beam-ACO in an iterative way for obtaining good SALBP-2 solutions. Our computational results show indeed that the proposed algorithm is currently a state-of-the-art method for the SALBP-2. It is able to generate optimal solutions, respectively the best upper bounds, for 283 out of 302 test cases. Moreover, in further 9 cases the algorithm is able to improve the currently best upper bounds.

Organization of the paper. In Section 2 we present a formal description of the tackled problem. Furthermore, in Section 3 the proposed algorithm is described. Finally, in Section 4 we present a detailed experimental evaluation and in Section 5 we conclude our work and offer an outlook to future work.

2. The SALBP-2

The SALBP-2 can formally be described as follows. An instance (T,G,m) consists of three components. $T = \{1,\ldots,n\}$ is a set of n tasks. Each task $i \in T$ has a pre-defined processing time $t_i > 0$. Without losing generality, the processing times are henceforth assumed to be integer values. Moreover, given is a precedence graph $G = (T,A)$, which is a directed, acyclic graph with T as node set. Finally, m is the pre-defined number of work stations which are ordered from 1 to m. An arc $l_{i,j} \in A$ indicates that $i \in T$ must be processed before $j \in T$. Given a task $j \in T$, $P_j \subset T$ denotes the set of tasks that must be processed before j. A feasible solution is obtained by assigning each task to exactly one work station such that the precedence constraints between the tasks are satisfied. The objective function consists in minimizing the so-called cycle time. The SALBP-2 can be expressed in the following way as an integer programming (IP) problem.

$$\min z \tag{1}$$

subject to:

$$\sum_{s=1}^{m} x_{is} = 1 \quad \forall i \in T \tag{2}$$

$$x_{is} \leq \sum_{s'=1}^{s} x_{js'} \quad \forall i \in T, s = 1,\ldots,m,\ j \in P_i \tag{3}$$

$$\sum_{i \in T} t_i x_{is} \leq z \quad s = 1,\ldots,m \tag{4}$$

$$x_{is} \in \{0,1\} \quad \forall i \in T, s = 1,\ldots,m \tag{5}$$

$$z > 0 \tag{6}$$

This IP model makes use of the following variables and constants: x_{is} is a binary variable which is set to 1 if and only if task $i \in T$ is assigned to work station $1 \leq s \leq m$. The objective function (1) minimizes the cycle time $z > 0$.[1] The constraints (2) ensure that each task $i \in T$ is assigned to a single work station $1 \leq s \leq m$. Constraints (3) reflect the precedence relationships between the tasks. More specifically, if task $i \in T$ is assigned to a work station $1 \leq s \leq m$, all tasks $j \in P_i$ must be assigned to work stations $1 \leq s' \leq m$ with $s' \leq s$. The constraints (4) ensure that the sum of the processing times of the tasks assigned to a work station $1 \leq s \leq m$ do not exceed the cycle time z.

Note that this model was chosen in order to present an easily understandable problem description. An evaluation of alternative models can be found in Pastor and Ferrer (2009), Pastor et al. (2007).

3. Iterative beam search

As mentioned in the introduction, the basic component of our algorithm for the SALBP-2 consists of beam search (BS), which is an incomplete derivative of branch & bound. BS was used for the first time in the context of speech recognition Lowerre (1976). Concerning combinatorial optimization problems, BS has especially been used for solving scheduling problems; see, for example, Ghirardi and Potts (2005), Ow and Morton (1988), Sabuncuoglu and Bayiz (1999), Valente and Alves (2005). To date only very few applications to other types of problems exist. Examples can be found in Akeba et al. (2009), Blum et al. (2009), Lee and Woodruff (2004). In the following we briefly describe how one of the standard variants of BS works. The crucial aspect of BS is the parallel extension of partial solutions in several ways. At all times, the algorithm keeps a set B of at most k_{bw} partial solutions, where B is the so-called *beam*, and k_{bw} is known as the *beam width*. At each step, at most k_{ext} feasible extensions of each partial solution in B are selected on the basis of greedy information. In general, this selection

1. Note that we refer to the variable cycle time of the IP model as z, while fixed cycle times are denoted by C.

is done deterministically. At the end of each step, the algorithm creates a new beam B by choosing up to k_{bw} partial solutions from the set of selected feasible extensions. For that purpose, BS algorithms determine – in the case of minimization – a lower bound value for each extension. Only the maximally k_{bw} best extensions – with respect to these lower bound values – are included in B. Finally, if any complete solution was generated, the algorithm returns the best of those. Note that the underlying constructive heuristic that defines feasible extensions of partial solutions and the lower bound function for evaluating partial solutions are crucial for the working of BS.

In the following, after describing the chosen solution representation, we first present a description of the implementation of the BS component, before we describe the algorithmic scheme in which this BS component is used.

3.1. Solution representation

The following solution representation is used by the proposed BS algorithm. A solution $\mathcal{S}$ is an ordered list $\mathcal{S} = \langle S_1, \ldots, S_m \rangle$ of m sets of tasks, where S_i denotes the set of tasks that are assigned to the i-th work station. Abusing notation we henceforth call S_i a work station. Note that for a solution $\mathcal{S}$ to be valid the following conditions must be fulfilled:

1. $\bigcup_{i=1}^{m} S_i = T = \{1, \ldots, n\}$ and $S_i \cap S_{i'} = \emptyset$ for $i = 1, \ldots, m$ and $i' = i+1, \ldots, m$. That is, each task is assigned to exactly one work station.
2. For each task $j \in S_i$ it is must hold that $P_j \subseteq \bigcup_{k=1}^{i} S_k$. This ensures that the precedence constraints between the tasks are not violated.

In this context it is also convenient to introduce the concept of the *reverse problem instance*. More specifically, the reverse problem instance (T, G^r, m) with respect to an original instance (T, G, m) is obtained by inverting the direction of all arcs of G. It is well-known from the literature Scholl and Becker (2006) that tackling the reverse problem instance may lead an exact algorithm faster to an optimal solution, respectively, may provide a better heuristic solution when tackled with the same heuristic as the original problem instance. Moreover, a solution $\mathcal{S}^r$ to the reverse problem instance (T, G^r, m) can easily be converted into a solution $\mathcal{S}$ to the original problem instance (T, G, m) as follows:

$$S_i := S^r_{m-i+1} \quad \text{for } i = 1, \ldots, m \tag{7}$$

3.2. The beam search component

The BS component described in this section – see Algorithm 1 for the pseudo-code – is the main component of the proposed algorithm for the SALBP-2. The algorithm requires a problem instance (T, G, m), a fixed cycle time C, a beam width k_{bw}, and a maximal

Algorithm 1 Beam search

1: **input:** an instance (T, G, m), a fixed cycle time C, a beam width k_{bw}, and k_{ext}
2: $d := 0$
3: Initialization of an empty solution $\mathcal{S}$
4: $B := \{\mathcal{S}\}$
5: $B_{\text{compl}} := \emptyset$
6: **while** $B \neq \emptyset$ **do**
7: $\quad B_{\text{ext}} := \emptyset$
8: $\quad d := d + 1$
9: $\quad$ **for all** $\mathcal{S} \in B$ **do**
10: $\quad\quad$ **for** $i = 1, \ldots, k_{\text{ext}}$ **do**
11: $\quad\quad\quad \mathcal{S}' := \mathcal{S}$ {copy partial solution $\mathcal{S}$ into $\mathcal{S}'$}
12: $\quad\quad\quad S'_d :=$ ExtendPartialSolution($\mathcal{S}', d, C$) {see Algorithm 2}
13: $\quad\quad\quad$ **if** solution $\mathcal{S}'$ is complete (that is, all tasks are assigned) **then**
14: $\quad\quad\quad\quad B_{\text{compl}} := B_{\text{compl}} \cup \{\mathcal{S}'\}$
15: $\quad\quad\quad$ **else**
16: $\quad\quad\quad\quad$ **if** $d < m$ and S'_d is different to the d-th work station of all other $\mathcal{S} \in B_{\text{ext}}$ **then**
17: $\quad\quad\quad\quad\quad B_{\text{ext}} := B_{\text{ext}} \cup \{\mathcal{S}'\}$
18: $\quad\quad\quad\quad$ **end if**
19: $\quad\quad\quad$ **end if**
20: $\quad\quad$ **end for**
21: $\quad$ **end for**
22: $\quad B \leftarrow$SelectSolutions(B_{ext}, k_{bw})
23: **end while**
24: **output:** If $B_{\text{compl}} \neq \emptyset$ the output is TRUE, otherwise FALSE

number of extensions k_{ext} as input. Given a fixed cycle time C and m (the number of work stations) BS tries to find at least one feasible solution. As mentioned before, the crucial aspect of BS is the extension of partial solutions in several possible ways. At each step the algorithm extends each partial solution from B in a limited number of ways. More specifically, given a partial solution with $d - 1 < m$ work stations already filled, an extension is generated by assigning a set of so-far unassigned tasks to the next work station S_d such that the given cycle time C is not surpassed and the precedence constraints between the tasks are respected (see lines 11–12 of Algorithm 1). The algorithm produces extensions in a (partially) probabilistic way rather than in the usual deterministic manner.[2] Each generated extension (partial solution) is either stored in set B_{compl} in case it is a complete solution, or in set B_{ext} otherwise (see lines 13–19 of

2. This is done for avoiding a *choice-without-replacement* process for which all possible work station fillings must be generated beforehand.

Algorithm 1). However, a partial solution is only stored in set B_{ext} if it uses at most $m-1$ work stations. Moreover, for a partial solution to be stored in set B_{ext} it is required that its d-th work station is different to the d-th work station of all partial solutions that are already in B_{ext}. This criterion can be seen as a computationally cheap, approximate way of checking if two partial solutions are equal. On the downside, with this criterion partial solutions might be excluded from further examination even though they do not belong to B_{ext}. Finally, BS creates a new beam B by selecting up to k_{bw} solutions from set B_{ext} of further extensible partial solutions (see line 22 of Algorithm 1). This is done in function SelectSolutions(B_{ext},k_{bw}) on the basis of a lower bound function LB$(\cdot)$. In the following we describe in detail the extension of partial solutions and the working of function SelectSolutions(B_{ext},k_{bw}).

Extending partial solutions. The generation of an extension of a partial solution $\mathcal{S}'$ with $d-1$ work stations already filled works as follows. Unassigned tasks are iteratively assigned to work station S'_d until the sum of their processing times is such that no other task can be added to S'_d without exceeding the given cycle time C. This procedure is pseudo-coded in Algorithm 2. At each step, T' denotes the set of so-far unassigned tasks that may be added to S'_d without violating any constraints. The definition of this set of *available tasks* is given in line 4, respectively 9, of Algorithm 2.

Algorithm 2 Function ExtendPartialSolution($\mathcal{S}',d,C$) of Algorithm 1

1: **input:** A partial solution $\mathcal{S}'$, the index d of the work station to be filled, and the cycle time C
2: $S'_d := \emptyset$
3: $c_{\text{rem}} := C$
4: $T' := \{i \in T \mid i \notin \bigcup_{r=1}^{d} S'_r, P_i \subseteq \bigcup_{r=1}^{d} S'_r, t_i \leq c_{\text{rem}}\}$
5: **while** $T' \neq \emptyset$ **do**
6: $\quad j :=$ChooseTask(T',C,c_{rem})
7: $\quad c_{\text{rem}} := c_{\text{rem}} - t_j$
8: $\quad S'_d := S'_d \cup \{j\}$
9: $\quad T' := \{i \in T \mid i \notin \bigcup_{r=1}^{d} S'_r, P_i \subseteq \bigcup_{r=1}^{d} S'_r, t_i \leq c_{\text{rem}}\}$
10: **end while**
11: **output:** Filled work station S'_d

It remains to describe the implementation of function ChooseTask(T',C,c_{rem}) of Algorithm 2. For that purpose let us first define the following subset of T':

$$T^{\text{sat}} := \{i \in T' \mid t_i = c_{\text{rem}}\} \tag{8}$$

This definition is such that T^{sat} contains all tasks that *saturate*, in terms of processing time, the d-th work station S_d. The choice of a task from T' is made on the basis of

greedy information, that is, on the basis of values $\eta_i > 0$ that are assigned to all tasks $i \in T'$ by a greedy function. The first action for choosing a task from T' consists in flipping a coin for deciding if the choice is made deterministically, or probabilistically. In case of a deterministic choice, there are two possibilities. First, if $T^{\text{sat}} \neq \emptyset$, the best task from T^{sat} is chosen, that is, the task with maximal greedy value among all tasks in T^{sat}. Otherwise, we choose the task with maximal greedy value from T'. In case of a probabilistic decision, a task from T' is chosen on the basis of the following probability distribution:

$$\mathbf{p}(i) := \frac{\eta_i}{\sum_{j \in T'} \eta_j} \quad , \forall i \in T' \tag{9}$$

For completing the description of function ChooseTask(T',C,c_{rem}), we must describe the definition of the greedy values η_i, $\forall i \in T$. In a first step a term γ_i is defined as follows:

$$\gamma_i := \kappa_1 \cdot \left(\frac{t_i}{C}\right) + \kappa_2 \cdot \left(\frac{|\text{Suc}_i^{\text{all}}|}{\max_{1 \leq j \leq n} \left|\text{Suc}_j^{\text{all}}\right|}\right) \quad , \forall i \in T \tag{10}$$

Hereby, $\text{Suc}_i^{\text{all}}$ denotes the set of all tasks that can be reached from i in precedence graph G via a directed path. This definition combines two greedy function that are often used in the context of assembly line balancing problems. The first one concerns the task processing times and the second one concerns the size of $\text{Suc}_i^{\text{all}}$. The influence of both heuristics can be adjusted via the setting of weights κ_1 and κ_2. In order to be more flexible we decided to allow for both weights a value from $[-1,1]$. This means that we consider for each heuristic potentially also its negation. This is motivated by experience from the field of scheduling, where some problem instances are more successfully solved by doing exactly the opposite of what is suggested by certain greedy functions. Given the γ_i-values, the greedy values η_i are then derived as follows:

$$\eta_i := \frac{\gamma_i - \gamma_{\min} + 1}{\gamma_{\max}} \quad \forall\, i \in T \quad , \tag{11}$$

where $\gamma_{\min}$, respectively $\gamma_{\max}$, denote the minimum, respectively maximum, values of all γ_i. Interestingly, for obtaining well-working greedy values, parameters κ_1 and κ_2 have to be chosen in a problem-instance-dependent way. A study concerning the values of parameters κ_1 and κ_2 is presented in Section 4.2.

The lower bound function. The new beam B is – at each step – chosen from B_{ext}. This choice is implemented by function SelectSolutions(B_{ext},k_{bw}) of Algorithm 2. First, the solutions in B_{ext} are ranked with respect to increasing lower bound values LB($\cdot$). Then, the $\min\{k_{\text{bw}}, |B_{\text{ext}}|\}$ highest ranked partial solutions from B_{ext} are selected. Let us denote

by $\overline{T} \subseteq T$ the set of tasks that have not yet been assigned to work stations in partial solution $\mathcal{S}'$. Then:

$$\mathrm{LB}(\mathcal{S}') = \left\lceil \frac{\sum_{i \in \overline{T}} t_i}{C} \right\rceil \tag{12}$$

Note that this lower bound is inspired by splitting-based bounds for the one-dimensional bin packing problem.

Algorithm 3 Iterative beam search (IBS) for the SALBP-2

input: an instance (T, G, m)
$C :=$ DetermineStartingCycleTime()
$C' := C$
$k_{\text{bw}} := 5$, $k_{\text{ext}} := 2$
success := FALSE
while not *success* **do**
  success := BeamSearch($(T, G, m), C, k_{\text{bw}}, k_{\text{ext}}$) {original instance}
  if not *success* **then**
    success := BeamSearch($(T, G^r, m), C, k_{\text{bw}}, k_{\text{ext}}$) {reverse instance}
    if not *success* **then** $C := C + 1$ **end if**
  end if
end while
if $C > C'$ **then**
  $C := C - 1$
  stop := FALSE
  while not *stop* **do**
    success := FALSE
    while time limit not reached **and not** *success* **do**
      if within 5% of time limit **then** $k_{\text{bw}} := 10$, $k_{\text{ext}} := 5$ **else** $k_{\text{bw}} := 150$, $k_{\text{ext}} := 20$ **end if**
      success := BeamSearch($(T, G, m), C, k_{\text{bw}}, k_{\text{ext}}$) {original instance}
      if not *success* **then**
        success := BeamSearch($(T, G^r, m), C, k_{\text{bw}}, k_{\text{ext}}$) {reverse instance}
      end if
    end while
    if *success* **then** $C := C - 1$ **else** *stop* := TRUE **end if**
  end while
  $C := C + 1$
end if
output: cycle time C

3.3. The algorithmic scheme

The BS component outlined in the previous section is used by an iterative algorithmic scheme that is presented in Algorithm 3. Henceforth this algorithmic scheme is labelled iterated beam search (IBS). The first step consists in determining a starting cycle time C, which is computed in funcion DetermineStartingCycleTime() of Algorithm 3 as

$$C := \max \left\{ \max_{i \in T} \{t_i\}, \left\lceil \frac{\sum_{i \in T} t_i}{m} \right\rceil \right\} \quad . \tag{13}$$

The algorithm works in two phases. In the first phase (see lines 4-12 of Algorithm 3) the algorithm tries to quickly find a first cycle time C for which a valid solution can be found. For this purpose BS is applied with the setting $k_{\text{bw}} = 5$ and $k_{\text{ext}} = 2$. Note that this setting was chosen after tuning by hand. Moreover, note that the first phase only takes a fraction of a second of computation time. This holds for all instances considered in Section 4. The second phase of the algorithm iteratively tries to find a valid solution for the next smaller cycle time. In this phase, the algorithm disposes over a certain time limit for each considered cycle time. Remember that the working of BS is partially probabilistic. Therefore, BS can repeatedly be applied to the same instance with potentially different outcomes. The first five percent of the above-mentioned time limit are spent by BS applications that use the setting $k_{\text{bw}} := 10$ and $k_{\text{ext}} := 5$. This is done with the intention of not wasting too much computation time, if not necessary. However, if BS is not able to solve the given cycle time with this setting, the remaining 95% of the available time are spent by BS applications using the setting $k_{\text{bw}} := 150$ and $k_{\text{ext}} := 20$. With this setting BS is considerably slower. However, the probability of finding feasible solutions is much higher than with the setting described before. The second phase of the algorithm ends when the time limit has passed without having found a feasible solution for the considered cycle time.

4. Experimental evaluation

IBS was implemented in ANSI C++, and GCC 3.4.0 was used for compiling the software. Experimental results were obtained on a PC with an AMD64X2 4400 processor and 4 Gb of memory. In the following we first describe the set of benchmark instances that we used for the experimental evaluation. Subsequently we present a study concerning some of the parameters of the proposed algorithm. Finally, the experimental results are presented.

4.1. Benchmark instances

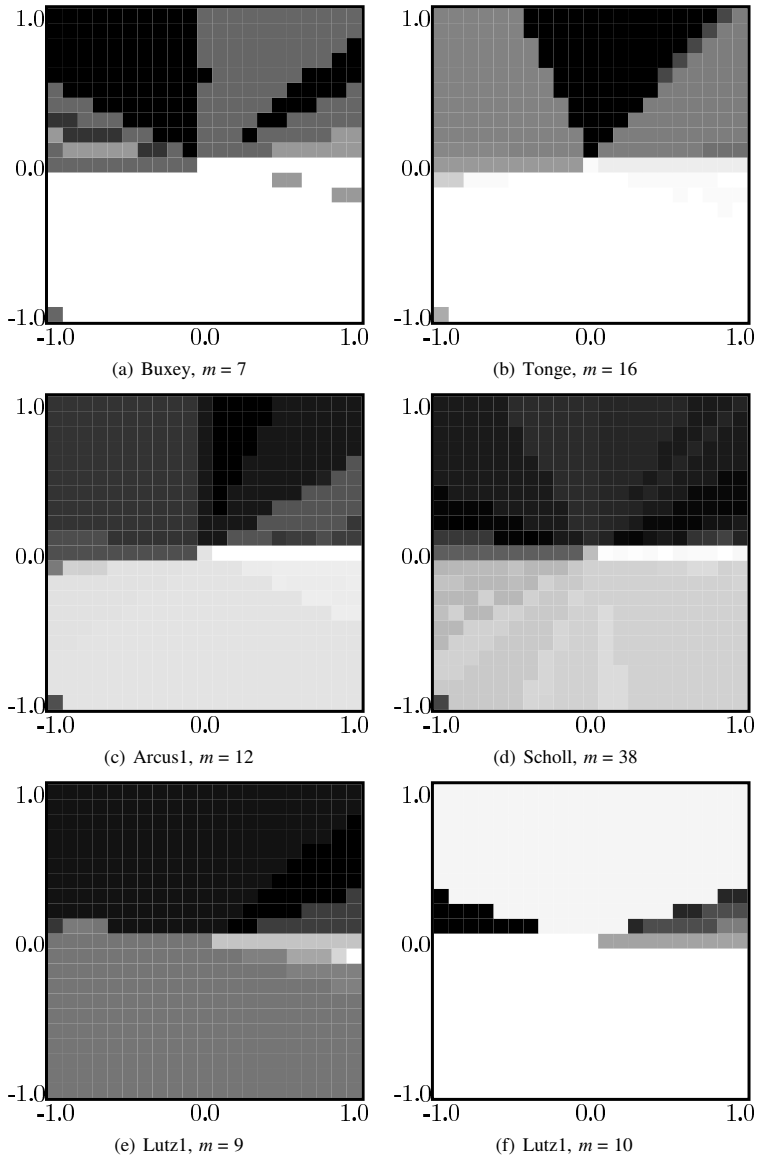
We used the usual set of 302 benchmark instances from the literature. They can be obtained – together with information about optimal solutions, respectively lower and upper bounds – from a website especially dedicated to all kind of assembly line balancing problems maintained by Armin Scholl, `http://www.assembly-line-balancing.de`. Each instance consists of a precedence graph G and a given number m of work stations. The benchmark set is composed of two subsets of instances, henceforth called Dataset1 and Dataset2. Dataset1 consists of 128 instances based on 9 different precedence graphs with a number of tasks between 29 to 111. Dataset2 is composed of 174 instances based on 8 different precedence graphs with a number of tasks varying from 53 to 297.

4.2. A study of parameters κ_1 and κ_2

During preliminar experiments we realized that parameters k_{bw} and k_{ext} have a rather low impact on the final results of IBS. In other words it is easy to find a reasonable setting for these parameters quite quickly. Their setting dynamically changes during a run of the algorithm as specified in Section 3.3. On the contrary, parameters κ_1 and κ_2 (see Eq. 10) have a rather high impact on the algorithms' performance. Remember that κ_1 is the weight of the greedy function concerning the task processing times, while κ_2 is the weight of the greedy function concerning the number of tasks that have to be processed after the task under consideration. As mentioned before, for both parameters we allowed values from $[-1,1]$. Instead of trying to find a good parameter setting for each single instance, we decided for a process aimed at identifying a single setting of κ_1 and κ_2 for all instances concerning the same precedence graph. For that purpose we applied a specific version of IBS for all combinations of $\kappa_1,\kappa_2 \in \{-1.0,-0.9,\ldots,0.0,\ldots,0.9,1.0\}$ to all 302 instances. This makes a total of 441 different settings for each instance. The specific version of IBS that we apply in the following differs from IBS as outlined in Algorithm 3 in that lines 14-24 are replaced by a single, deterministic, application of beam search with $k_{bw} = 150$ and $k_{ext} = 20$. This was done for the purpose of saving computation time. Based on the obtained results we chose the settings presented in Table 1

Table 1: *Values of parameters κ_1 and κ_2 for the final experiments.*

Graph	κ_1	κ_2	**Graph**	κ_1	κ_2
Arcus1	0.0	1.0	Lutz2	−0.1	0.9
Arcus2	−0.5	0.9	Lutz3	0.0	1.0
Barthol2	0.0	1.0	Mukherje	0.0	1.0
Barthold	0.0	1.0	Sawyer	0.0	1.0
Buxey	0.0	1.0	Scholl	0.0	1.0
Gunther	−0.4	0.8	Tonge	−0.1	0.2
Hahn	0.0	1.0	Warnecke	0.0	1.0
Kilbridge	0.0	1.0	Wee−Mag	0.0	1.0
Lutz1	−0.1	0.9			



(a) Buxey, $m = 7$ (b) Tonge, $m = 16$

(c) Arcus1, $m = 12$ (d) Scholl, $m = 38$

(e) Lutz1, $m = 9$ (f) Lutz1, $m = 10$

Figure 1: *Results presented in graphical form for six representative instances. The y-axis ranges over the values of κ_1, while the x-axis ranges over the values of κ_2. The gray levels of the squares indicate the quality of the algorithm when run with the corresponding setting: the lighter a square is painted, the better is the parameter setting.*

for the different precedence graphs. It is interesting to note that, apart from a few exceptions, the greedy heuristic based on task processing times does not seem necessary for obtaining good results.

In Figure 1 a representative sample of the results is provided in graphical form. The y-axis of the presented graphics varies over the different values of κ_1, while the x-axis

ranges over the allowed values of κ_2. Note that each graphic consists of 441 squares representing the 441 different combinations of values for κ_1 and κ_2. The gray level in which each square is painted indicates the quality of the algorithm when run with the corresponding parameter setting. In particular, black color denotes the worst setting, whereas white color indicates the best algorithm setting. In some cases such as (Buxey, $m = 7$) and (Tonge, $m = 16$), as shown in Figures 1(a) and 1(b), there is a wide range of good settings, which are basically all those with $\kappa_1 \leq 0$. In other examples such as (Arcus1, $m = 12$) and (Scholl, $m = 38$) it is strictly required to set κ_1 to 0 and κ_2 to a positive value for obtaining good solutions; see Figures 1(c) and 1(d). Finally, the graphics shown in Figures 1(e) and 1(f) indicate that even for the same precedence graph a good parameter setting might depend strongly on the number of work stations.

Table 2: *Differences in algorithm performance when considering the best and the worst parameter setting.*

Instance	Best setting	Worst setting	Difference (%)
(Buxey, $m = 7$)	47	52	10.64
(Tonge, $m = 16$)	222	265	19.37
(Arcus1, $m = 12$)	12599	13767	9.27
(Scholl, $m = 38$)	1857	2031	9.37
(Lutz1, $m = 9$)	1637	1801	10.02
(Lutz1, $m = 10$)	1525	1619	6.16

It is also interesting to quantify the differences in algorithm performance for different parameter settings. Table 2 shows for the six cases presented in Figure 1 the result of the algorithm with the best setting (column **Best setting**), the result of the algorithm with the worst setting (column **Worst setting**), and the difference (in percent) between these two settings. The results in Table 2 show that there are considerable differences in performance between the best and the worst algorithm setting. This underlines the importance of finding opportune values for κ_1 and κ_2.

4.3. Results

Algorithm IBS was applied 20 times to all 302 instances. Herefore we used a computation time limit of 180 seconds for each cycle time, that is, IBS was given maximally 180 seconds for finding a feasible solution for a given cycle time. In case of success, the algorithm has again 180 seconds for the next smaller cycle time, etc. Detailed results of IBS for all 302 instances are given in Tables 7 and 8 that are to be found in Appendix A. The data are, in both tables, presented as follows. The first two columns provide the name of the precedence graph and the number of work stations (m). The third column (labelled *bub*) provides the values of the optimal solutions in case they are known. If they are unknown the column reports the currently best upper bound. In case a value is not proved to be optimal it is overlined. More in detail, in 15 out of 302 cases optimality

has not been proved yet. The remaining five columns are reserved for the results of IBS. The first of these five columns contains the value of the best solution found by IBS over 20 runs. In case this value is presented with a gray background, a new best upper bound has been found. On the other side, if this value is marked by an asterisk, the obtained result does not reach the value of the best upper bound. In all other cases the values correspond to the values of the best upper bounds. The second column provides the average over 20 runs, while the third column contains the corresponding standard deviation. The fourth column gives the average time (in seconds) at which the best solution of a run was found, averaged over 20 runs. The fifth column provides the corresponding standard deviation. From the results presented in Tables 7 and 8 (see Appendix A) we can observe that IBS obtains optimal solutions, respectively best upper bounds, in 276 out of 302 cases. Moreover, new best upper bounds are obtained in 6 cases. This is remarkable as – despite a considerable amount of ongoing research – in the last 14 years no improved solutions have been reported. Only in 20 cases (all concerning precedence graphs Arcus1, Arcus2, Scholl, and Warnecke) our algorithm was not able to find the best solutions known. However, in most of these cases the deviation from the best upper bound is no more than one unit of cycle time.

In addition to Tables 7 and 8 the results of IBS are presented in a summarized way in Table 3, in comparison to three other algorithms. TABUSEARCH Scholl and Voss (1996), even though already published in 1996, still counts as the current state-of-the-art heuristic method for SALBP-2. DE_RKS is the best version of a differential evolution (DE) algorithm proposed in Nearchou (2007), and PNA-FOR is a Petri net-based heuristic published in Kilincci (2010). The last two methods are, to our knowledge, the most recently published heuristic methods for SALBP-2. Three measures are used in Table 3 for the comparison of IBS with these three algorithms. The row labelled **#opt** provides the number of best upper bounds found by each method (over 302). Moreover, the row labelled **mrd (%)** gives the *mean relative deviation (in percent)* of the results obtained by the four algorithms from the best-known upper bounds for all 302 instances. Finally, row **time** contains the average computation time of the algorithms for all 302 instances. Concerning the quality of the results, we can conclude that IBS clearly outperforms its competitors. For the correct interpretation of the computation times it has to be taken into account that the four algorithms were executed on computers with very different processor speeds. While TABUSEARCH was executed on a 80486 DX2-66 processor, PNA-FOR was run on an Athlon XP 2000+ processor with

Table 3: *Results of* IBS *in comparison to the best (*TABUSEARCH*), respectively most recent (*DE_RKS *and* PNA-FOR*), methods from the literature.*

	DE_RKS	PNA-FOR	TABUSEARCH	**Iterative Beam Search** (IBS)
#opt	n/g	39	168	282
mrd (%)	2.64	2.73	0.40	0.0029
time (s)	10.74	407.28	84.90	31.61

Note: n/g means **not given**

1.67 GHz, and DE_RKS was run on a Pentium IV processor with 1.7 GHz. This means that TABUSEARCH was run by far on the slowest machine, IBS by far on the fastest machine, and PNA-FOR and DE_RKS on comparable machines. Given the computation times in Table 3 we can savely conclude that TABUSEARCH is the fastest algorithm, and PNA-FOR is the slowest one. However, note that assembly line balancing is, in most cases, not a time-critical application. In other words, for most practical purposes it does not matter if an algorithm takes 1 minute or 6 hours of computation time.

In the following we present the results of IBS in comparison to DE_RKS and PNA-FOR in the same way as done in Nearchou (2007) and Kilincci (2010). In these works, results were presented as averages over instances based on the same precedence graph, and also averaged over Dataset1 and Dataset2. The quality of the results is given in terms of the *mean relative deviation (in percent)* from the best-known upper bounds. Tables 4 and 5 clearly show that IBS is largely superior to both competitor algorithms.

Table 4: *Results of* IBS *in comparison to* DE_RKS *and* PNA-FOR *for the 128 instances of* Dataset1 *(averaged over precedence graphs)*

Graph	DE_RKS		PNA-FOR	**Iterative Beam Search** (IBS)	
	mrd (%)	time (s)	mrd (%)	mrd (%)	time (s)
Buxey	1.16	0.80	3.07	0.0	0.06
Sawyer	2.27	1.64	4.00	0.0	0.14
Lutz1	0.32	0.88	4.09	0.0	0.55
Gunther	0.14	1.08	1.27	0.0	0.02
Kilbridge	0.66	1.43	2.19	0.0	0.0067
Tonge	1.88	3.71	2.53	0.0	7.73
Arcus1	0.99	5.29	2.47	0.0287	152.34
Lutz2	3.08	1.00	2.99	0.0	0.035
Arcus2	4.96	19.02	2.06	0.0066	135.83
Average:	**1.72**	**3.87**	**2.57**	**0.0058**	**51.76**

Table 5: *Results of* IBS *in comparison to* DE_RKS *and* PNA-FOR *for the 174 instances of* Dataset2 *(averaged over precedence graphs).*

Graph	DE_RKS		PNA-FOR	**Iterative Beam Search** (IBS)	
	mrd (%)	time (s)	mrd (%)	mrd (%)	time (s)
Hahn	0.0	1.00	2.52	0.0	0.065
Warnecke	3.74	3.53	5.57	0.0579	1.54
Wee-Mag	1.23	3.68	1.56	0.0	0.65
Lutz3	1.68	5.62	2.59	0.0	0.61
Mukherje	n/a	n/a	1.04	0.0	2.52
Barthold	0.26	19.47	1.02	0.0	0.079
Barthol2	6.85	33.08	3.97	0.0	0.96
Scholl	9.51	43.95	3.21	0.0028	98.69
Average:	**3.32**	**15.79**	**2.85**	**0.00071**	**16.79**

Table 6: *Results of a high-performance version of* IBS.

Graph	m	Result	Graph	m	Result
Arcus1 (83)	8	9554	Arcus2 (111)	23	6560
	11	7085*		24	6282
	12	6412		25	6101
	17	4527*		26	5855
	18	4323*	Scholl (297)	31	2247
	19	4071*		36	1936*
	20	3886*		42	1660*
Arcus2 (111)	14	10747		46	1515
	15	10035		47	1484*
	16	9413*		49	1423
	17	8857*	Warnecke (58)	25	64
	18	8377	Wee-Mag (75)	18	87
	19	7922		19	85
	20	7524		23	67
	21	7187		27	65
	22	6856		28	64

4.4. Results of a high-performance version

In an attempt to further improve on the results of our algorithm we decided to apply a high-performance version of IBS to all problem instances for which the optimal solution is unknown and, additionally, to all instances where IBS – with the settings as outlined in the previous section – was not able to find the best known upper bounds. This high-performance version is obtained as follows. First, 1800 seconds are used as a time limit for each cycle time. Second, in line 17 of Algorithm 3 only 1% of the time limit is used (instead of 5%). Third, for each application of beam search in lines 18 and 20 of Algorithm 3 the beam width k_{bw} is randomly chosen from $[150,250]$ and the number of extensions is randomly chosen from $[20,40]$. Moreover, with a probability of 0.5 the heuristic information is – for each application of beam search – calculated using the weight values as outlined in Table 1. Otherwise, the weight values are chosen randomly from $[-1,1]$. With these modifications we applied IBS exactly once to all the instances of Table 6. The results of the algorithm are given in column **Result**: In case the result value is presented with a gray background, a new best upper bound has been found. On the other hand, if this value is marked by an asterisk, the obtained result is inferior to the value of the best known upper bound. Indeed, the number of instances for which the best-known upper bound can not be found before is reduced from 20 to 10 instances. Moreover, the algorithm is now able to find new best-known upper bounds in 9 (instead of only 6) cases. Summarizing, this amounts to 283 best-known upper bounds found and 9 new best upper bounds obtained. In one case, (Scholl, $m = 49$), the new upper bound is provenly optimal, as its value coincides with the best known lower bound.

5. Conclusions and future work

In this work we have proposed an iterative beam search algorithm for the simple assembly line balancing problem with a fixed number of work stations, SALBP-2. The experimental evalution of the algorithm has shown that it is currently a state-of-the-art method for this problem. Appart from producing optimal solutions, respectively best upper bounds, for 283 out of 302 test cases, our algorithm generated new best-known upper bounds in further 9 test cases. Encouraged by the results for the SALBP-1 variant of the problem – as published in Blum (2008) – and the results obtained in this paper for the SALBP-2 we intent to apply similar algorithms based on beam search to other assembly line balancing problems.

Acknowledgements

This work was supported by grant TIN2007-66523 (FORMALISM) of the Spanish government. Moreover, Christian Blum acknowledges support from the *Ramón y Cajal* program of the Spanish Ministry of Science and Innovation.

Many thanks go to Armin Scholl for verifying the new best-known solutions found by the algorithm proposed in this work. Finally, we would also like to express our thanks to Cristóbal Miralles who was involved as a co-author of a similar work for the more general assembly line worker assignment and balancing problem.

References

H. Akeba, M. Hifib and R. MHallah (2009). A beam search algorithm for the circular packing problem. *Computers & Operations Research*, 36(5), 1513–1528.

E. J. Anderson and M. C. Ferris (1994). Genetic algorithms for combinatorial optimization: The assembly line balancing problem. *ORSA Journal on Computing*, 6, 161–173.

C. Blum (2008). Beam-ACO for simple assembly line balancing. *INFORMS Journal on Computing*, 20(4), 618–627.

C. Blum and C. Miralles (2011). On solving the assembly line worker assignment and balancing problem via beam search. *Computers & Operations Research*, 38(1), 328–339.

C. Blum, M. J. Blesa and M. López Ibáñez (2009). Beam search for the longest common subsequence problem. *Computers & Operations Research*, 36(12), 3178–3186.

W.-C. Chiang (1998). The application of a tabu search metaheuristic to the assembly line balancing problem. *Annals of Operations Research*, 77, 209–227.

M. Ghirardi and C. N. Potts (2005). Makespan minimization for scheduling unrelated parallel machines: A recovering beam search approach. *European Journal of Operational Research*, 165(2), 457–467.

S. Gosh and R. J. Gagnon (1989). A comprehensive literature review and analysis of the design, balancing and scheduling of assembly systems. *International Journal of Production Research*, 27, 637–670.

S. T. Hackman, M. J. Magazine and T. S. Wee (1989). Fast, effective algorithms for simple assembly line balancing problems. *Operations Research*, 37, 916–924.

A. Henrici (1994). A comparison between simulated annealing and tabu search with an example from the production planning. In H. Dyckhoff et al., editor, *Operations Research Proceedings*, 498–503. Springer Verlag, Berlin, Germany.

O. Kilincci (2010). A Petri net-based heuristic for simply assembly line balancing problem of type 2. *International Journal of Advanced Manufacturing Technology*, 46, 329–338.

R. Klein and A. Scholl (1996). Maximizing the production rate in simple assembly line balancing – A branch and bound procedure. *European Journal of Operational Research*, 91, 367–385.

G.-C. Lee and D. L. Woodruff (2004). Beam search for peak alignment of NMR signals. *Analytica Chimica Acta*, 513(2), 413–416.

B. Lowerre (1976). *The Harpy Speech Recognition System*. PhD thesis, Carnegie Mellon University.

A. C. Nearchou (2007). Balancing large assembly lines by a new heuristic based on differential evolution. *International Journal of Advanced Manufacturing Technology*, 34, 1016–1029.

P. S. Ow and T. E. Morton (1988). Filtered beam search in scheduling. *International Journal of Production Research*, 26, 297–307.

R. Pastor and L. Ferrer (2009). An improved mathematical program to solve the simple assembly line balancing problem. *International Journal of Production Research*, 47(11), 2943–2959.

R. Pastor, L. Ferrer and A. García (2007). Evaluating optimization models to solve SALBP. In O. Gervasi and M. L. Gavrilova, editors, *Proceedings of ICCSA 2007 – International Conference on Computational Science and Its Applications*, volume 4705 of *Lecture Notes in Computer Science*, 791–803. Springer Verlag, Berlin, Germany.

I. Sabuncuoglu and M. Bayiz (1999). Job shop scheduling with beam search. *European Journal of Operational Research*, 118, 390–412.

M. E. Salveson (1955). The assembly line balancing problem. *Journal of Industrial Engineering*, 6, 18–25.

A. Scholl (1994). Ein B&B-Verfahren zur Abstimmung von Einprodukt-Fließbändern bei gegebener Stationsanzahl. In H. Dyckhoff et al., editor, *Operations Research Proceedings*, 175–181. Springer Verlag, Berlin, Germany.

A. Scholl (1999). *Balancing and sequencing assembly lines*. Physica Verlag, Heidelberg, Germany, 2nd edition edition.

A. Scholl and C. Becker (2006). State-of-the-art exact and heuristic solution procedures for simple assembly line balancing. *European Journal of Operational Research*, 168(3), 666–693.

A. Scholl and S. Voss (1996). Simple assembly line balancing – Heuristic approaches. *Journal of Heuristics*, 2, 217–244.

H. F. Ugurdag, R. Rachamadugu and C. A. Papachristou (1997). Designing paced assembly lines with fixed number of stations. *European Journal of Operational Research*, 102, 488–501.

J. M. S. Valente and R. A. F. S. Alves (2005). Filtered and recovering beam search algorithms for the early/tardy scheduling problem with no idle time. *Computers & Industrial Engineering*, 48(2), 363–375.

T. Watanabe, Y. Hashimoto, L. Nishikawa and H. Tokumaru (1995). Line balancing using a genetic evolution model. *Control Engineering Practice*, 3, 69–76.

Appendix A

Table 7: *Detailed results of* IBS *for 302 test instances (Part A).*

Graph	m	bub	Iterative Beam Search (IBS)					Graph	m	bub	Iterative Beam Search (IBS)				
			best	avg	std	time (s)	std				best	avg	std	time (s)	std
Arcus1 (83)	3	25236	25236	25236.00	(0.00)	0.56	(0.29)	Barthold (148)	9	626	626	626.00	(0.00)	0.13	(0.03)
	4	18927	18927	18927.00	(0.00)	6.16	(3.92)		10	564	564	564.00	(0.00)	0.10	(0.02)
	5	15142	15142	15142.00	(0.00)	80.08	(16.69)		11	513	513	513.00	(0.00)	0.09	(0.00)
	6	12620	12620	12620.00	(0.00)	15.52	(5.76)		12	470	470	470.00	(0.00)	0.09	(0.00)
	7	10826	10826	10826.00	(0.00)	59.80	(19.89)		13	434	434	434.00	(0.00)	0.16	(0.02)
	8	9554	*9555	9555.55	(0.51)	120.54	(63.16)		14	403	403	403.00	(0.00)	0.14	(0.01)
	9	8499	8499	8500.70	(0.80)	130.73	(117.23)		15	383	383	383.00	(0.00)	0.05	(0.00)
	10	7580	7580	7580.95	(0.60)	311.89	(109.87)	Buxey (29)	7	47	47	47.00	(0.00)	0.46	(0.51)
	11	7084	*7086	7086.70	(0.47)	71.13	(52.77)		8	41	41	41.00	(0.00)	0.00	(0.00)
	12	6412	*6413	6414.10	(0.72)	362.32	(66.26)		9	37	37	37.00	(0.00)	0.01	(0.00)
	13	5864	5864	5864.00	(0.00)	129.18	(12.99)		10	34	34	34.00	(0.00)	0.00	(0.00)
	14	5441	5441	5441.00	(0.00)	2.02	(0.41)		11	32	32	32.00	(0.00)	0.00	(0.00)
	15	5104	5104	5104.45	(0.60)	222.29	(73.92)		12	28	28	28.00	(0.00)	0.01	(0.00)
	16	4850	4850	4850.00	(0.00)	33.84	(9.59)		13	27	27	27.00	(0.00)	0.01	(0.00)
	17	4516	*4524	4526.20	(0.89)	282.49	(100.96)		14	25	25	25.00	(0.00)	0.00	(0.00)
	18	4317	*4322	4323.20	(0.77)	291.73	(111.23)	Gunther (35)	6	84	84	84.00	(0.00)	0.00	(0.00)
	19	4068	*4073	4074.80	(1.01)	326.78	(125.49)		7	72	72	72.00	(0.00)	0.03	(0.00)
	20	3882	*3886	3889.25	(2.05)	594.21	(204.09)		8	63	63	63.00	(0.00)	0.01	(0.00)
	21	3691	3691	3691.00	(0.00)	5.58	(0.81)		9	54	54	54.00	(0.00)	0.05	(0.04)
	22	3691	3691	3691.00	(0.00)	0.02	(0.00)		10	50	50	50.00	(0.00)	0.01	(0.00)
Arcus2 (111)	3	50133	50133	50133.00	(0.00)	0.09	(0.03)		11	48	48	48.00	(0.00)	0.00	(0.00)
	4	37600	37600	37600.00	(0.00)	0.41	(0.02)		12	44	44	44.00	(0.00)	0.01	(0.00)
	5	30080	30080	30080.00	(0.00)	0.43	(0.22)		13	42	42	42.00	(0.00)	0.01	(0.00)
	6	25067	25067	25067.00	(0.00)	1.13	(0.59)		14	40	40	40.00	(0.00)	0.02	(0.00)
	7	21486	21486	21486.00	(0.00)	1.76	(1.04)		15	40	40	40.00	(0.00)	0.01	(0.00)
	8	18800	18800	18800.00	(0.00)	11.36	(2.86)	Hahn (53)	3	4787	4787	4787.00	(0.00)	0.00	(0.00)
	9	16711	16711	16711.00	(0.00)	28.03	(12.51)		4	3677	3677	3677.00	(0.00)	0.08	(0.01)
	10	15040	15040	15040.00	(0.00)	38.67	(15.08)		5	2823	2823	2823.00	(0.00)	0.01	(0.00)
	11	13673	13673	13673.00	(0.00)	49.47	(12.83)		6	2400	2400	2400.00	(0.00)	0.05	(0.01)
	12	12534	12534	12534.00	(0.00)	43.34	(15.17)		7	2336	2336	2336.00	(0.00)	0.29	(0.01)
	13	11570	11570	11570.00	(0.00)	32.60	(14.63)		8	1907	1907	1907.00	(0.00)	0.09	(0.03)
	14	10747	*10748	10748.00	(0.00)	62.69	(16.17)		9	1827	1827	1827.00	(0.00)	0.00	(0.00)
	15	10035	*10036	10036.40	(0.50)	112.54	(63.37)		10	1775	1775	1775.00	(0.00)	0.00	(0.00)
	16	9412	*9416	9416.60	(0.68)	290.35	(90.83)	Kilbridge (45)	3	184	184	184.00	(0.00)	0.00	(0.00)
	17	8855	*8864	8864.90	(0.31)	87.41	(52.86)		4	138	138	138.00	(0.00)	0.01	(0.00)
	18	$\overline{8377}$	8377	8377.00	(0.00)	8.11	(1.00)		5	111	111	111.00	(0.00)	0.00	(0.00)
	19	$\overline{7928}$	7924	7925.60	(0.60)	205.09	(76.12)		6	92	92	92.00	(0.00)	0.00	(0.00)
	20	$\overline{7526}$	7524	7524.40	(0.50)	159.72	(57.92)		7	79	79	79.00	(0.00)	0.01	(0.00)
	21	$\overline{7188}$	*7192	7193.40	(0.82)	318.78	(89.64)		8	69	69	69.00	(0.00)	0.01	(0.00)
	22	$\overline{6859}$	6858	6858.20	(0.41)	226.94	(75.52)		9	62	62	62.00	(0.00)	0.01	(0.00)
	23	$\overline{6561}$	6560	6563.10	(1.45)	428.68	(168.53)		10	56	56	56.00	(0.00)	0.01	(0.00)
	24	$\overline{6289}$	6284	6285.65	(1.46)	311.77	(144.55)		11	55	55	55.00	(0.00)	0.01	(0.00)
	25	$\overline{6106}$	*6112	6114.15	(0.99)	305.50	(90.99)	Lutz1 (32)	8	1860	1860	1860.00	(0.00)	0.14	(0.00)
	26	$\overline{5856}$	*5858	5860.45	(1.76)	661.48	(192.47)		9	1638	1638	1638.00	(0.00)	2.41	(0.34)
	27	5689	5689	5689.00	(0.00)	9.29	(0.17)		10	1526	1526	1526.00	(0.00)	0.18	(0.01)
Barthol2 (148)	27	157	157	157.00	(0.00)	0.31	(0.03)		11	1400	1400	1400.00	(0.00)	0.01	(0.00)
	28	152	152	152.00	(0.00)	0.44	(0.07)		12	1400	1400	1400.00	(0.00)	0.00	(0.00)
	29	146	146	146.00	(0.00)	0.80	(0.32)	Lutz2 (89)	9	54	54	54.00	(0.00)	0.02	(0.01)
	30	142	142	142.00	(0.00)	0.15	(0.03)		10	49	49	49.00	(0.00)	0.03	(0.00)
	31	137	137	137.00	(0.00)	0.67	(0.09)		11	45	45	45.00	(0.00)	0.04	(0.00)
	32	133	133	133.00	(0.00)	0.57	(0.07)		12	41	41	41.00	(0.00)	0.03	(0.00)
	33	129	129	129.00	(0.00)	0.52	(0.06)		13	38	38	38.00	(0.00)	0.03	(0.00)
	34	125	125	125.00	(0.00)	0.47	(0.05)		14	35	35	35.00	(0.00)	0.03	(0.00)
	35	121	121	121.00	(0.00)	0.94	(0.15)		15	33	33	33.00	(0.00)	0.03	(0.00)
	36	118	118	118.00	(0.00)	0.87	(0.11)		16	31	31	31.00	(0.00)	0.02	(0.00)
	37	115	115	115.00	(0.00)	0.92	(0.19)		17	29	29	29.00	(0.00)	0.04	(0.00)
	38	112	112	112.00	(0.00)	0.76	(0.06)		18	28	28	28.00	(0.00)	0.04	(0.00)
	39	109	109	109.00	(0.00)	0.70	(0.13)		19	26	26	26.00	(0.00)	0.04	(0.01)
	40	106	106	106.00	(0.00)	0.91	(0.32)		20	25	25	25.00	(0.00)	0.04	(0.00)
	41	104	104	104.00	(0.00)	0.72	(0.05)		21	24	24	24.00	(0.00)	0.02	(0.00)
	42	101	101	101.00	(0.00)	2.77	(5.43)		22	23	23	23.00	(0.00)	0.04	(0.00)
	43	99	99	99.00	(0.00)	0.84	(0.23)		23	22	22	22.00	(0.00)	0.05	(0.00)
	44	97	97	97.00	(0.00)	0.59	(0.15)		24	21	21	21.00	(0.00)	0.05	(0.00)
	45	95	95	95.00	(0.00)	0.76	(0.25)		25	20	20	20.00	(0.00)	0.02	(0.00)
	46	93	93	93.00	(0.00)	0.68	(0.29)		26	19	19	19.00	(0.00)	0.05	(0.02)
	47	91	91	91.00	(0.00)	0.67	(0.15)		27	19	19	19.00	(0.00)	0.03	(0.00)
	48	89	89	89.00	(0.00)	1.17	(0.47)		28	18	18	18.00	(0.00)	0.05	(0.00)
	49	87	87	87.00	(0.00)	0.83	(0.26)	Lutz3 (89)	3	548	548	548.00	(0.00)	0.01	(0.00)
	50	85	85	85.90	(0.31)	3.28	(10.33)		4	411	411	411.00	(0.00)	0.03	(0.00)
	51	84	84	84.00	(0.00)	2.60	(2.09)		5	329	329	329.00	(0.00)	0.01	(0.00)
Barthold (148)	3	1878	1878	1878.00	(0.00)	0.01	(0.00)		6	275	275	275.00	(0.00)	0.01	(0.00)
	4	1409	1409	1409.00	(0.00)	0.02	(0.00)		7	236	236	236.00	(0.00)	0.02	(0.00)
	5	1127	1127	1127.00	(0.00)	0.02	(0.00)		8	207	207	207.00	(0.00)	0.04	(0.00)
	6	939	939	939.00	(0.00)	0.06	(0.01)		9	184	184	184.00	(0.00)	5.54	(4.14)
	7	805	805	805.00	(0.00)	0.11	(0.00)		10	165	165	165.00	(0.00)	0.09	(0.01)
	8	705	705	705.00	(0.00)	0.04	(0.01)		11	151	151	151.00	(0.00)	0.06	(0.00)

Table 8: *Detailed results of* IBS *for 302 test instances (Part B).*

Graph	m	bub	Iterative Beam Search (IBS)					Graph	m	bub	Iterative Beam Search (IBS)				
			best	avg	std	time (s)	std				best	avg	std	time (s)	std
Lutz3 (89)	12	138	138	138.00	(0.00)	0.02	(0.00)	Tonge (70)	6	585	585	585.00	(0.00)	0.03	(0.02)
	13	128	128	128.00	(0.00)	0.05	(0.03)		7	502	502	502.00	(0.00)	0.06	(0.01)
	14	118	118	118.00	(0.00)	0.67	(0.52)		8	439	439	439.00	(0.00)	0.04	(0.01)
	15	110	110	110.00	(0.00)	0.80	(0.66)		9	391	391	391.00	(0.00)	0.05	(0.00)
	16	105	105	105.00	(0.00)	0.34	(0.30)		10	352	352	352.00	(0.00)	0.06	(0.01)
	17	98	98	98.00	(0.00)	0.19	(0.05)		11	320	320	320.00	(0.00)	0.06	(0.02)
	18	93	93	93.00	(0.00)	0.16	(0.05)		12	294	294	294.00	(0.00)	0.09	(0.01)
	19	89	89	89.00	(0.00)	0.05	(0.03)		13	271	271	271.00	(0.00)	0.06	(0.01)
	20	85	85	85.00	(0.00)	0.14	(0.05)		14	251	251	251.80	(0.41)	20.25	(45.26)
	21	80	80	80.00	(0.00)	0.11	(0.02)		15	235	235	235.00	(0.00)	0.35	(0.16)
	22	76	76	76.00	(0.00)	4.20	(3.30)		16	221	221	221.00	(0.00)	0.97	(0.93)
	23	74	74	74.00	(0.00)	0.29	(0.18)		17	208	208	208.00	(0.00)	1.02	(0.95)
Mukherje (94)	3	1403	1403	1403.00	(0.00)	0.01	(0.00)		18	196	196	196.15	(0.37)	46.76	(39.05)
	4	1052	1052	1052.00	(0.00)	0.10	(0.00)		19	186	186	186.35	(0.49)	87.09	(64.62)
	5	844	844	844.00	(0.00)	0.02	(0.00)		20	177	177	177.90	(0.31)	9.46	(39.39)
	6	704	704	704.00	(0.00)	0.02	(0.00)		21	170	170	170.00	(0.00)	8.71	(9.43)
	7	621	621	621.00	(0.00)	0.12	(0.00)		22	162	162	162.00	(0.00)	2.52	(2.45)
	8	532	532	532.00	(0.00)	0.09	(0.01)		23	156	156	156.00	(0.00)	0.11	(0.04)
	9	471	471	471.00	(0.00)	0.09	(0.01)		24	156	156	156.00	(0.00)	0.02	(0.00)
	10	424	424	424.00	(0.00)	0.02	(0.00)		25	156	156	156.00	(0.00)	0.02	(0.00)
	11	391	391	391.00	(0.00)	0.10	(0.01)	Warnecke (58)	3	516	516	516.00	(0.00)	0.02	(0.00)
	12	358	358	358.00	(0.00)	0.05	(0.01)		4	387	387	387.00	(0.00)	0.08	(0.08)
	13	325	325	325.00	(0.00)	0.17	(0.06)		5	310	310	310.00	(0.00)	0.03	(0.00)
	14	311	311	311.00	(0.00)	0.08	(0.01)		6	258	258	258.00	(0.00)	0.10	(0.06)
	15	288	288	288.00	(0.00)	0.09	(0.02)		7	222	222	222.00	(0.00)	0.03	(0.00)
	16	268	268	268.00	(0.00)	0.20	(0.02)		8	194	194	194.00	(0.00)	0.05	(0.01)
	17	251	251	251.00	(0.00)	0.45	(0.05)		9	172	172	172.00	(0.00)	2.40	(2.28)
	18	239	239	239.00	(0.00)	0.24	(0.02)		10	155	155	155.00	(0.00)	0.05	(0.01)
	19	226	226	226.00	(0.00)	0.03	(0.01)		11	142	142	142.00	(0.00)	0.04	(0.00)
	20	220	220	220.05	(0.22)	57.07	(43.23)		12	130	130	130.00	(0.00)	0.07	(0.01)
	21	208	208	208.00	(0.00)	0.12	(0.00)		13	120	120	120.00	(0.00)	0.08	(0.01)
	22	200	200	200.00	(0.00)	0.13	(0.06)		14	111	111	111.00	(0.00)	0.29	(0.24)
	23	189	189	189.00	(0.00)	0.12	(0.02)		15	104	104	104.00	(0.00)	0.27	(0.13)
	24	179	179	179.00	(0.00)	0.48	(0.41)		16	98	98	98.00	(0.00)	0.08	(0.01)
	25	172	172	172.00	(0.00)	0.43	(0.18)		17	92	92	92.00	(0.00)	0.26	(0.14)
	26	171	171	171.00	(0.00)	0.27	(0.03)		18	87	87	87.00	(0.00)	0.27	(0.14)
Sawyer (30)	7	47	47	47.00	(0.00)	0.14	(0.11)		19	84	84	84.00	(0.00)	0.10	(0.06)
	8	41	41	41.00	(0.00)	0.06	(0.03)		20	79	79	79.00	(0.00)	0.13	(0.03)
	9	37	37	37.00	(0.00)	0.03	(0.01)		21	76	76	76.00	(0.00)	0.08	(0.04)
	10	34	34	34.00	(0.00)	0.02	(0.00)		22	73	73	73.00	(0.00)	0.53	(0.36)
	11	31	31	31.00	(0.00)	0.82	(0.85)		23	69	69	69.00	(0.00)	18.10	(21.25)
	12	28	28	28.00	(0.00)	0.03	(0.01)		24	66	66	66.90	(0.31)	2.88	(9.10)
	13	26	26	26.00	(0.00)	0.02	(0.00)		25	64	*65	65.00	(0.00)	12.30	(11.87)
	14	25	25	25.00	(0.00)	0.02	(0.00)		26	64	64	64.00	(0.00)	0.05	(0.01)
Scholl (297)	25	2787	2787	2787.00	(0.00)	37.11	(6.11)		27	60	60	60.00	(0.00)	1.61	(0.73)
	26	2680	2680	2680.00	(0.00)	32.56	(4.71)		28	59	59	59.00	(0.00)	0.16	(0.06)
	27	2580	2580	2580.00	(0.00)	58.06	(8.65)		29	56	56	56.00	(0.00)	1.43	(1.15)
	28	2488	2488	2488.00	(0.00)	53.78	(12.43)	Wee-Mag (75)	3	500	500	500.00	(0.00)	0.01	(0.00)
	29	2402	2402	2402.00	(0.00)	45.57	(9.93)		4	375	375	375.00	(0.00)	0.03	(0.00)
	30	2322	2322	2322.30	(0.47)	96.86	(53.36)		5	300	300	300.00	(0.00)	0.02	(0.00)
	31	2247	*2248	2248.00	(0.00)	51.23	(4.32)		6	250	250	250.00	(0.00)	0.03	(0.00)
	32	2177	2177	2177.50	(0.51)	91.50	(63.17)		7	215	215	215.00	(0.00)	0.01	(0.00)
	33	2111	2111	2111.80	(0.41)	57.19	(39.95)		8	188	188	188.00	(0.00)	0.04	(0.01)
	34	2049	2049	2049.20	(0.41)	117.47	(53.03)		9	167	167	167.00	(0.00)	0.01	(0.00)
	35	1991	1991	1991.00	(0.00)	95.54	(17.19)		10	150	150	150.00	(0.00)	0.08	(0.02)
	36	1935	*1936	1936.00	(0.00)	61.22	(8.78)		11	137	137	137.00	(0.00)	0.06	(0.01)
	37	1883	1883	1883.05	(0.22)	140.57	(34.46)		12	125	125	125.00	(0.00)	0.10	(0.02)
	38	1834	1834	1834.00	(0.00)	125.92	(34.53)		13	116	116	116.00	(0.00)	0.02	(0.00)
	39	1787	1787	1787.35	(0.49)	192.12	(55.85)		14	108	108	108.00	(0.00)	0.54	(0.60)
	40	1742	1742	1742.00	(0.00)	153.02	(29.66)		15	100	100	100.00	(0.00)	0.24	(0.19)
	41	1700	1700	1700.00	(0.00)	81.51	(18.06)		16	94	94	94.00	(0.00)	0.05	(0.01)
	42	1659	*1660	1660.00	(0.00)	146.58	(32.11)		17	89	89	89.00	(0.00)	0.42	(0.28)
	43	1621	1621	1621.00	(0.00)	149.08	(21.75)		18	$\overline{87}$	87	87.00	(0.00)	0.07	(0.01)
	44	1584	1584	1584.15	(0.37)	127.52	(54.06)		19	$\overline{85}$	85	85.00	(0.00)	0.08	(0.02)
	45	1549	1549	1549.00	(0.00)	150.65	(32.59)		20	77	77	77.00	(0.00)	0.13	(0.04)
	46	1515	*1516	1516.00	(0.00)	121.28	(20.06)		21	72	72	72.00	(0.00)	0.08	(0.01)
	47	1483	*1484	1484.00	(0.00)	165.44	(28.64)		22	69	69	69.00	(0.00)	0.06	(0.01)
	48	1452	1452	1452.70	(0.47)	151.98	(60.76)		23	$\overline{67}$	67	67.00	(0.00)	4.34	(3.58)
	49	$\overline{1427}$	1424	1424.00	(0.00)	113.46	(22.47)		24	66	66	66.00	(0.00)	0.25	(0.14)
	50	1394	1394	1394.85	(0.37)	143.63	(53.48)		25	65	65	65.00	(0.00)	11.36	(26.58)
	51	1386	1386	1386.00	(0.00)	2.23	(0.09)		26	65	65	65.00	(0.00)	0.05	(0.01)
	52	1386	1386	1386.00	(0.00)	0.17	(0.01)		27	$\overline{65}$	65	65.00	(0.00)	0.02	(0.00)
Tonge (70)	3	1170	1170	1170.00	(0.00)	0.01	(0.00)		28	$\overline{64}$	64	64.00	(0.00)	0.02	(0.00)
	4	878	878	878.00	(0.00)	0.02	(0.00)		29	63	63	63.00	(0.00)	0.02	(0.00)
	5	702	702	702.00	(0.00)	0.05	(0.02)		30	56	56	56.00	(0.00)	0.07	(0.00)