The Pareto IV power series cure rate model with applications

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Additional material

The hessian matrix for this model is given by

$$H(\xi) = \begin{pmatrix} H_{\beta\beta} & H_{\beta\lambda} \\ H_{\beta\lambda}^{\intercal} & H_{\lambda\lambda} \end{pmatrix},$$

where

$$\begin{aligned}
H_{\beta\beta} = \sum_{i=1}^{n} x_i^{\intercal} x_i \theta_i (1-\eta\theta_i) \Bigg\{ & (1-\delta_i)\frac{S_i}{A(\theta_i S_i)}\Bigg[A''(\theta_i S_i)\theta_i(1-\eta\theta_i)S_i - \frac{[A'(\theta_i S_i)]^2}{A(\theta_i S_i)}\theta_i(1-\eta\theta_i)S_i \\
& + A'(\theta_i S_i)(1-2\eta\theta_i)\Bigg] + \delta_i\Bigg[-\eta + \frac{S_i}{A'(\theta_i S_i)}\Bigg(A'''(\theta_i S_i)\theta_i(1-\eta\theta_i)S_i \\
& - \frac{[A''(\theta_i S_i)]^2}{A'(\theta_i S_i)}\theta_i(1-\eta\theta_i)S_i + A''(\theta_i S_i)(1-2\eta\theta_i)\Bigg)\Bigg] - \frac{1}{A(\theta_i)}\Bigg(A''(\theta_i)\theta_i(1-\eta\theta_i) \\
& - \frac{[A'(\theta_i)]^2}{A(\theta_i)}\theta_i(1-\eta\theta_i) + A'(\theta_i)(1-2\eta\theta_i)\Bigg)\Bigg\}
\end{aligned}$$

$$\begin{aligned}
H_{\lambda\lambda} = \sum_{i=1}^{n} \Bigg\{ & (1-\delta_i)\frac{\theta_i}{A(\theta_i S_i)}\Bigg[A''(\theta_i S_i)\theta_i\left(\frac{\partial S_i}{\partial\lambda}\right)^{\intercal}\left(\frac{\partial S_i}{\partial\lambda}\right) - \frac{[A'(\theta_i S_i)]^2}{A(\theta_i S_i)}\theta_i\left(\frac{\partial S_i}{\partial\lambda}\right)^{\intercal}\left(\frac{\partial S_i}{\partial\lambda}\right) \\
& + A'(\theta_i S_i)\frac{\partial^2 S_i}{\partial\lambda^{\intercal}\partial\lambda}\Bigg] + \delta_i\Bigg[\frac{\partial^2 \log f_i}{\partial\lambda^{\intercal}\partial\lambda} + \frac{\theta_i}{A'(\theta_i S_i)}\Bigg(A'''(\theta_i S_i)\theta_i\left(\frac{\partial S_i}{\partial\lambda}\right)^{\intercal}\left(\frac{\partial S_i}{\partial\lambda}\right) \\
& - \frac{[A''(\theta_i S_i)]^2}{A'(\theta_i S_i)}\theta_i\left(\frac{\partial S_i}{\partial\lambda}\right)^{\intercal}\left(\frac{\partial S_i}{\partial\lambda}\right) + A''(\theta_i S_i)\frac{\partial^2 S_i}{\partial\lambda\partial\lambda^{\intercal}}\Bigg)\Bigg]\Bigg\}
\end{aligned}$$

$$\begin{aligned}
H_{\beta\lambda} = \sum_{i=1}^{n} x_i^{\intercal}\theta_i(1-\eta\theta_i)\Bigg\{ & (1-\delta_i)\frac{1}{A(\theta_i S_i)}\Bigg[A''(\theta_i S_i)\theta_i S_i\frac{\partial S_i}{\partial\lambda} - \frac{[A'(\theta_i S_i)]^2}{A(\theta_i S_i)}\theta_i S_i\frac{\partial S_i}{\partial\lambda} \\
& + A'(\theta_i S_i)\frac{\partial S_i}{\partial\lambda}\Bigg] + \delta_i\Bigg[\frac{1}{A'(\theta_i S_i)}\Bigg(A'''(\theta_i S_i)\theta_i S_i\frac{\partial S_i}{\partial\lambda} - \frac{[A''(\theta_i S_i)]^2}{A'(\theta_i S_i)}\theta_i S_i\frac{\partial S_i}{\partial\lambda} \\
& + A''(\theta_i S_i)\frac{\partial S_i}{\partial\lambda}\Bigg)\Bigg]\Bigg\},
\end{aligned}$$

where

$$\eta = \begin{cases} 1, & \text{for Logarithmic and Negative Binomial models,} \\ 0, & \text{for Poisson and Binomial models,} \end{cases} \tag{1}$$

The first, second and third derivatives of $A(\cdot)$ function are presented in Table 1 for each particular model.

Table 1: *Derivates of* $A(\theta)$.

Distribution	$A(\theta)$	$A'(\theta)$	$A''(\theta)$	$A'''(\theta)$
Poisson	e^{θ}	e^{θ}	e^{θ}	e^{θ}
Logarithmic	$-\frac{\log(1-\theta)}{\theta}$	$\frac{\theta+(1-\theta)\log(1-\theta)}{(1-\theta)\theta^2}$	$\frac{\theta(3\theta-2)-2(1-\theta)^2\log(1-\theta)}{(1-\theta)^2\theta^3}$	$\frac{6(1-\theta)^3\log(1-\theta)+\theta(11\theta^2-15\theta+6)}{\theta^4(1-\theta)^3}$
Negative Binomial	$(1-\theta)^{-q}$	$q(1-\theta)^{-(q+1)}$	$-q(q+1)(1-\theta)^{-(q+2)}$	$q(q+1)(q+2)(1+\theta)^{-(q-3)}$
Binomial	$(1+\theta)^{q}$	$q(1+\theta)^{q-1}$	$q(q-1)(1+\theta)^{q-2}$	$q(q-1)(q-2)(1+\theta)^{q-3}$

On the other hand, the first and second derivatives of f_i and S_i in relation to $\lambda=(\sigma,\gamma,\alpha)$ are

$$\frac{\partial S_i}{\partial \alpha} = -\left(1+\left(\frac{t_i}{\sigma}\right)^{1/\gamma}\right)^{-\alpha} \times \log\left(1+\left(\frac{t_i}{\sigma}\right)^{1/\gamma}\right)$$

$$\frac{\partial S_i}{\partial \sigma} = \frac{\alpha}{\gamma\sigma}\left(1+\left(\frac{t_i}{\sigma}\right)^{1/\gamma}\right)^{-\alpha-1}\left(\frac{t_i}{\sigma}\right)^{1/\gamma}$$

$$\frac{\partial S_i}{\partial \gamma} = \frac{\alpha}{\gamma^2} \times \left(1+\left(\frac{t_i}{\sigma}\right)^{1/\gamma}\right)^{-\alpha-1} \times \left(\frac{t_i}{\sigma}\right)^{1/\gamma} \times \log\left(\frac{t_i}{\sigma}\right)$$

$$\frac{\partial^2 S_i}{\partial \alpha^2} = \left(1+\left(\frac{t_i}{\sigma}\right)^{1/\gamma}\right)^{-\alpha} \times \log^2\left(1+\left(\frac{t_i}{\sigma}\right)^{1/\gamma}\right)$$

$$\frac{\partial^2 S_i}{\partial \alpha \partial \sigma} = \frac{1}{\gamma\sigma} \times \left(1+\left(\frac{t_i}{\sigma}\right)^{1/\gamma}\right)^{-\alpha-1} \times \left(\frac{t_i}{\sigma}\right)^{1/\gamma} \times \left(1-\alpha\log\left(1+\left(\frac{t_i}{\sigma}\right)^{1/\gamma}\right)\right)$$

$$\frac{\partial^2 S_i}{\partial \alpha \partial \gamma} = -\frac{1}{\gamma^2} \times \left(1+\left(\frac{t_i}{\sigma}\right)^{1/\gamma}\right)^{-\alpha-1} \times \left(\frac{t_i}{\sigma}\right)^{1/\gamma} \times \log\left(\frac{t_i}{\sigma}\right) \times \left(1-\alpha\log\left(1+\left(\frac{t_i}{\sigma}\right)^{1/\gamma}\right)\right)$$

$$\frac{\partial^2 S_i}{\partial \sigma^2} = \frac{\alpha}{\gamma^2\sigma^2} \times \left(1+\left(\frac{t_i}{\sigma}\right)^{1/\gamma}\right)^{-\alpha-2} \times \left(\frac{t_i}{\sigma}\right)^{1/\gamma} \times \left((\alpha-\gamma)\left(\frac{t_i}{\sigma}\right)^{1/\gamma} - (\gamma+1)\right)$$

$$\frac{\partial^2 S_i}{\partial \sigma \partial \gamma} = \left(1+\left(\frac{t_i}{\sigma}\right)^{1/\gamma}\right)^{-\alpha-2} \times \left(\frac{t_i}{\sigma}\right)^{1/\gamma} \times \left(\left(\frac{t_i}{\sigma}\right)^{1/\gamma} \times \left(\alpha^2\log\left(\frac{t_i}{\sigma}\right)/(\gamma^3\sigma)\right.\right.$$
$$\left.\left. - \alpha/(\gamma^2\sigma)\right) - \alpha\log\left(\frac{t_i}{\sigma}\right)/(\gamma^3\sigma) - \alpha/(\gamma^2\sigma)\right)$$

$$\frac{\partial^2 S_i}{\partial \gamma^2} = \left(1+\left(\frac{t_i}{\sigma}\right)^{1/\gamma}\right)^{-\alpha-2} \times \left(\frac{t_i}{\sigma}\right)^{1/\gamma} \times \left(\left(\frac{t_i}{\sigma}\right)^{1/\gamma} \times \left(\alpha^2\left(\log\left(\frac{t_i}{\sigma}\right)\right)^2/\gamma^4 - 2\alpha\log\left(\frac{t_i}{\sigma}\right)/\gamma^3\right)\right.$$
$$\left. - \alpha\log^2\left(\frac{t_i}{\sigma}\right)/\gamma^4 - 2\alpha\log\left(\frac{t_i}{\sigma}\right)/\gamma^3\right)$$

$$\frac{\partial^2 \log f_i}{\partial \alpha^2} = -\frac{1}{\alpha^2}$$

$$\frac{\partial^2 \log f_i}{\partial \alpha \partial \sigma} = \frac{1}{\gamma\sigma} \times \left(\frac{t_i}{\sigma}\right)^{1/\gamma} \Big/ \left(1 + \left(\frac{t_i}{\sigma}\right)^{1/\gamma}\right)$$

$$\frac{\partial^2 \log f_i}{\partial \alpha \partial \gamma} = \frac{1}{\gamma^2} \times \left(\frac{t_i}{\sigma}\right)^{1/\gamma} \times \log\left(\frac{t_i}{\sigma}\right) \times \left(1 + \left(\frac{t_i}{\sigma}\right)^{1/\gamma}\right)^{-1}$$

$$\frac{\partial^2 \log f_i}{\partial \sigma^2} = -\frac{1}{\gamma^2\sigma^2} \times \left(\alpha\gamma\left(\frac{t_i}{\sigma}\right)^{2/\gamma} + \left(\frac{t_i}{\sigma}\right)^{1/\gamma} \times (\alpha(\gamma+1) - \gamma + 1) - \gamma\right) \times \left(1 + \left(\frac{t_i}{\sigma}\right)^{1/\gamma}\right)^{-2}$$

$$\frac{\partial^2 \log f_i}{\partial \sigma \partial \gamma} = -\frac{1}{\gamma^3\sigma} \times \left(\alpha\gamma\left(\frac{t_i}{\sigma}\right)^{2/\gamma} + \left(\frac{t_i}{\sigma}\right)^{1/\gamma} \times \left((\alpha+1)\log\left(\frac{t_i}{\sigma}\right) + \gamma(\alpha-1)\right) - \gamma\right)$$

$$\left(1 + \left(\frac{t_i}{\sigma}\right)^{1/\gamma}\right)^{-2}$$

$$\frac{\partial^2 \log f_i}{\partial \gamma^2} = -\frac{1}{\gamma^4} \times \left(1 + \left(\frac{t_i}{\sigma}\right)^{1/\gamma}\right)^{-2} \times \left(\gamma\left(\frac{t_i}{\sigma}\right)^{2/\gamma} \times \left(2\log(\sigma) + 2(\alpha+1)\log\left(\frac{t_i}{\sigma}\right)\right.\right.$$

$$\left. - 2\log(t_i) - \gamma\right) + \left(\frac{t_i}{\sigma}\right)^{1/\gamma}\left(4\gamma\log(\sigma) + (\alpha+1)\log^2\left(\frac{t_i}{\sigma}\right) + 2\gamma(\alpha+1)\log\left(\frac{t_i}{\sigma}\right)\right.$$

$$\left.\left. - 2\gamma(2\log(t_i) + \gamma)\right) + \gamma(2\log(\sigma) - 2\log(t_i) - \gamma)\right)$$